Chapter 2

Fourier transformation of vector valued Fourier hyperfunctions

In this chapter, we study the Fourier transformation of vector valued Fourier hyperfunctions.

Here we use the notation of Part I and part II of this book. As for this, we refer the references Ito [65], [66].

2.1 Definition

In this section, we assume that $E$ is a Fréchet space. Then we introduce the notion of the Fourier transformation of $E$-valued Fourier hyperfunctions on $\mathbb{R}^n$ following Junker [26]. The Fourier transformation is an isomorphism of $\mathcal{B}(\mathbb{R}^n; E) = \mathcal{A}'(\mathbb{R}^n; E)$. Thereby, we can give the representation of an $E$-valued Fourier hyperfunction by a class of $E$-valued slowly increasing holomorphic functions on $\mathbb{C}^n \setminus \mathbb{R}^n$. Thus we give the explicit formula of operation of an element in $\mathcal{B}(\mathbb{R}^n; E)$ on $\mathcal{A}(\mathbb{R}^n)$.

In the following, we also use the notions and the notation in section 1.1.1 of Part III.

Definition 2.1.1 Let $T$ be an element in $\mathcal{B}(\mathbb{R}^n; E) = \mathcal{A}'(\mathbb{R}^n; E)$. Then we define the Fourier transformation $\mathcal{F}T$ of $T$ by the formula

$$\mathcal{F}T(\varphi) = T(\mathcal{F}^{-1}\varphi)$$
for any $\varphi \in A(\mathbb{R}^n)$.

We also define the inverse Fourier transformation $\hat{F}^{-1}T$ of $T$ by the formula

$$\hat{F}^{-1}T(\varphi) = T(F\varphi)$$

for any $\varphi \in A(\mathbb{R}^n)$.

Thus we have the equalities

$$\hat{F} = (F^{-1})^* \quad \text{and} \quad \hat{F}^{-1} = F^*.$$

Here $(F^{-1})^*$ and $F^*$ denote the dual mappings of $F^{-1}$ and $F$ respectively. Thus we have the equalities

$$F^*(F^{-1})^* = \text{the identity mapping of } \mathcal{B}(\mathbb{R}^n; E)$$

and

$$(F^{-1})^* = (F^*)^{-1}.$$ 

Hereafter we use the notation $\mathcal{F} = \hat{F} = (F^{-1})^*$ on $\mathcal{B}(\mathbb{R}^n; E)$ for simplicity.

Here, we prepare the notation used in this book.

If $A$ and $B$ are cones in $\mathbb{R}^n$, $A \subset\subset B$ means that the closure of $A$ has a compact neighborhood in the closure of $B$ with respect to the topology of $\mathbb{R}^n$.

We say that $\Gamma$ is a proper closed convex cone in $\mathbb{R}^n$ if it is a closed convex cone which does not contain any line in the whole.

For a proper closed convex cone $\Gamma$ in $\mathbb{R}^n$, $\Gamma^\circ$ is the polar set

$$\Gamma^\circ = \{ \xi \in \mathbb{R}^n; < x, \xi > \geq 0 \text{, for any } \xi \in \Gamma \}$$

of $\Gamma$.

For a $n$-tuple $\sigma = (\sigma_1, \cdots, \sigma_n) \in N = \{-1, 1\}^n$, $\Gamma_\sigma$ is the quadrant in $\mathbb{R}^n$ defined to be the cone

$$\Gamma_\sigma = \{ x \in \mathbb{R}^n; \sigma_j x_j > 0, j = 1, 2, \cdots, n \}.$$
Then \( K_\sigma \) is the closure of \( \Gamma_\sigma \) in \( \mathbb{R}^n \). \((\Gamma_\sigma^*)^i\) is the interior of the polar set \( \Gamma_\sigma^* \) of \( \Gamma_\sigma \).

Then we have the following.

**Theorem 2.1.2 (Junker)** We use the notation in the above. Every element \( T \in \mathcal{B}(\tilde{\mathbb{R}}^n; E) \) is decomposed as

\[
T = \sum_{\sigma \in N} T_\sigma, \ (T_\sigma \in \mathcal{A}(K_\sigma; E), \ \sigma \in N).
\]

Here we put \( N = \{-1, 1\}^n \).

**Proof** This is a special case of Ito [65], Theorem 5.1.6 Q.E.D.

### 2.2 The Paley-Wiener theorem

In this section, we prove the Paley-Wiener theorem for the Fourier transformation of vector valued Fourier hyperfunctions.

At first, we recall the definition of the sheaf of germs of \( E \)-valued slowly increasing holomorphic functions over \( \mathbb{C}^n \) following Juker[26].

**Definition 2.2.1 (the sheaf of germs of \( E \)-valued slowly increasing holomorphic functions)** We define \( E\tilde{\mathcal{O}} \) to be the sheafification of the presheaf \( \{\tilde{\mathcal{O}}(\Omega; E); \ \Omega \text{ is an open set in } \mathbb{C}^n\} \). Here, for an open set \( \Omega \) in \( \mathbb{C}^n \), the section module \( \tilde{\mathcal{O}}(\Omega; E) \) is the space of all \( E \)-valued holomorphic functions \( f(z) \) on \( \Omega \cap \mathbb{C}^n \) which satisfy the condition

\[
\sup_{z \in K \cap \mathbb{C}^n} \|f(z)\|e(-\varepsilon|z|) < \infty
\]

for any positive \( \varepsilon \) and any compact set \( K \) in \( \Omega \) and any seminorm \( \| \cdot \| \) defining the topology of \( E \).

**Remark** By the above definition, it is easy to see that \( E\tilde{\mathcal{O}}|\mathbb{C}^n = E\mathcal{O} \) holds.
Here we use the notation in section 1.1.

If we define \( V_0 = \mathbb{R}^n \times \sqrt{-1} \mathbb{R}^n \) and \( V_j = \mathbb{R}^n \times \sqrt{-1} \{ y \in \mathbb{R}^n; \ y_j \neq 0 \} \), and if we put \( \mathcal{V} = \{ V_j \}_{j=0}^n \) and \( \mathcal{V}' = \{ V_j \}_{j=1}^n \), then \((\mathcal{V}, \mathcal{V}')\) is an open covering of \((\tilde{C}^n, \tilde{C}^n \setminus \mathbb{R}^n)\). Then we can define the \( n \)-th relative cohomology group \( H^n(\mathcal{V}, \mathcal{V}', E \tilde{O}) \) of the covering \((\mathcal{V}, \mathcal{V}')\). We denote \( H^n(\mathcal{V}, \mathcal{V}', E \tilde{O}) \) by \( H^n \) for short.

Then we choose a function \( F_\sigma(\zeta) \in \tilde{O}(\mathbb{R}^n \times \sqrt{-1}(\Gamma_\sigma)^i; E) \) for \( \sigma \in N \).

For a function \( \{ F_\sigma; \ \sigma \in N \} \) of \( \tilde{O}(V_1 \cap V_2 \cap \cdots \cap V_n; E) \), we denote by \([F] = \{ F_\sigma; \ \sigma \in N \} \) the element of \( H^n \), one representative of which is that function \( \{ F_\sigma; \ \sigma \in N \} \). Then the class of \( E \)-valued slowly increasing holomorphic functions \([F] = \{ F_\sigma; \ \sigma \in N \} \) operates on \( \mathcal{A}(\mathbb{R}^n) \) continuously by the formula

\[
[F](f) = \sum_{\sigma \in N} (-1)^n \text{sign}(\sigma) \int F_\sigma(x + i\varepsilon)f(x + i\varepsilon)dx.
\]

If \( \{ F_\sigma; \ \sigma \in N \} \) is a function of \( \sum_{i \neq j} \tilde{O}(\bigcap V_i; E) \), we have

\[
[F](f) = 0
\]

by Cauchy’s theorem. Here, for \( \varepsilon_\sigma = (\varepsilon_1^\sigma, \ldots, \varepsilon_n^\sigma) \in \mathbb{R}^n \), \(|\varepsilon_j^\sigma|\)'s are sufficiently small but not zero. Then we put

\[
\text{sign}(\sigma) = \prod_{j=1}^n \sigma_j, \ \sigma_j = \text{sign}(\varepsilon_j^\sigma) = \frac{\varepsilon_j^\sigma}{|\varepsilon_j^\sigma|}, \ (j = 1, 2, \ldots, n).
\]

Thus \([F]\) defines an \( E \)-valued Fourier hyperfunction in \( \tilde{B}(\mathbb{R}^n; E) = \mathcal{A}'(\mathbb{R}^n; E) \).

**Theorem 2.2.2 (the Paley-Wiener theorem)** We use the notation in section 1.1. Let \( \Gamma \) be a proper closed convex cone in \( \mathbb{R}^n \) and let \( K \) be its closure in \( \mathbb{R}^n \). For the sake of simplicity, we assume that the vertex of the cone \( \Gamma \) be at the origin and it satisfies the condition \( \Gamma \subset \subset \{ x_1 \geq -\varepsilon \} \).
Choose $T \in \tilde{\mathcal{B}}(\tilde{\mathbb{R}}^n; \ E)$. Then $T \in \mathcal{A}^l(K; \ E)$ if and only if it satisfies the following conditions (1) and (2):

(1) $T(e(-i < z, \zeta >)) \in \mathcal{O}(\mathbb{R}^n \times \sqrt{-1}(\Gamma^o)^i; \ E)$ holds.

(2) For every $\Gamma' \subset \subset \Gamma^o$ and $\varepsilon > 0$, there exists some positive constant $C$ such that we have the estimate

$$
\|T(e(-i < z, \zeta >))\| \leq Ce(\varepsilon|\text{Re}\zeta| + \chi_{\Gamma', \varepsilon}(\text{Im}\zeta))
$$

for $\zeta \in \mathbb{R}^n \times \sqrt{-1}\Gamma'$. Here $\|\cdot\|$ means an arbitrary seminorm defining the topology of $E$ and we put

$$
\chi_{\Gamma', \varepsilon}(\eta) = \sup\{|x, \eta| + \varepsilon|x|; x \in \Gamma - \varepsilon(1, 0, \cdots, 0)\}.
$$

Proof. It goes in the same way as in the scalar valued case. So we omit the proof. Q.E.D.

As for Theorem 2.2.2, we refer to Junker [26].

**Definition 2.2.3** We use the notation is section 1.1. For $T \in \tilde{\mathcal{B}}(\tilde{\mathbb{R}}^n; \ E)$, we decompose $T$ as follows:

$$
T = \sum_{\sigma \in N} T_\sigma, \ (T_\sigma \in \mathcal{A}^l(K_\sigma; \ E), \ \sigma \in N).
$$

Putting

$$
F_\sigma(\zeta) = \text{sign}(\sigma)T_\sigma(e(-i < z, \zeta >))
$$

for $\text{Im}\zeta \in (\Gamma^o)^i$ for $\sigma \in N$, we define $\mathcal{F}_\sigma T = \{F_\sigma; \ \sigma \in N\} \in H^n$. We call this transformation $\mathcal{F}_\sigma$ the **Fourier-Carleman-Leray-Sato transformation**. We say that $\mathcal{F}_S$ is **FCLS transformation** for short.

Now we prove that $\mathcal{F}_S T$ is well-defined as element of $\tilde{\mathcal{B}}(\tilde{\mathbb{R}}^n; \ E)$. In fact, by the Paley-Wiener theorem, we have $F_\sigma(\zeta) \in \tilde{\mathcal{O}}(\tilde{\mathbb{R}}^n \times \sqrt{-1}(\Gamma_\sigma^o)^i; \ E)$. Thus the class of $E$-valued slowly increasing holomorphic functions $[F] = \{F_\sigma; \ \sigma \in N\}$ in $H^n$ operates on $\mathcal{A}(\tilde{\mathbb{R}}^n)$ continuously by the above formula (*).
But the class $[F]$ is independent of the decomposition of $T$ in Theorem 1.1.4. Indeed, the ambiguity of the decomposition of $T$ arises from an element

$$S \in \mathcal{A}'(K_\sigma; E) \cap \mathcal{A}'(K_\tau; E).$$

Then, by Theorem 5.1.4 of Part I of Ito [65], such an element $S$ can be considered as an element of $\mathcal{A}'(K_\sigma \cap K_\tau; E)$. Since $K_\sigma \cap K_\tau$ is contained in a proper closed convex cone of dimension less than $n$, whose vertex is at the origin, $S(e(-i < z, \zeta >))$ is slowly increasing and holomorphic in the whole complex plane with respect to at least one variable by virtue of the Paley-Wiener theorem. Hence such a function $S(e(-i < z, \zeta >))$ belongs to $\sum_{\sigma \in N} \mathcal{O}(\bigcap_{i \neq j} V_i; E)$ and operates on $\mathcal{A}(\tilde{\mathbb{R}}^n)$ as the zero linear mapping. Hence the ambiguity of the decomposition of $T$ does not affect $F_sT$. Thus $F_sT$ is well-defined.

Here we have the following.

**Theorem 2.2.4** We have $F_s = F$.

**Proof** Choose $T \in \mathcal{B}(\tilde{\mathbb{R}}^n; E)$. We put

$$F^{-1}T = S = \sum_{\sigma \in N} S_{\sigma}.$$  

Then, for $f \in \mathcal{A}(\tilde{\mathbb{R}}^n)$, we have the equality

$$F_S(F^{-1}T)(f) = \sum_{\sigma \in N} (-1)^n \int S_{\sigma}(e(-i < z, \zeta >)) f(\zeta) d\xi$$

$$= \left( \sum_{\sigma \in N} S_{\sigma} \right) \left( \int e(-i < z, \zeta >) f(\zeta) d\xi \right)$$

$$= S(Ff) = (F^{-1}T)(Ff) = T(f).$$

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Thus we have the equality
\[ \mathcal{F}_s(\mathcal{F}^{-1}T) = T \]
for every \( T \in \tilde{\mathcal{B}}(\mathbb{R}^n; E) \). Thus we have the equality \( \mathcal{F}_s \mathcal{F}^{-1} = \text{id} \). Hence we have the equality \( \mathcal{F}_s = \mathcal{F} \).

This completes the proof. Q.E.D.

Thus we have the following.

**Theorem 2.2.5** We use the notation in the above. Then we have the isomorphism \( \tilde{\mathcal{B}}(\mathbb{R}^n; E) \cong H^n \).

Thus we have defined the Fourier transform of an element of \( \tilde{\mathcal{B}}(\mathbb{R}^n; E) \) via “boundary values” of \( E \)-valued slowly increasing holomorphic functions in tubular domains. Thereby we can give the explicit formula (*) of operation of an \( E \)-valued Fourier hyper-function in \( \tilde{\mathcal{B}}(\mathbb{R}^n; E) \) on \( \mathcal{A}(\mathbb{R}^n) \).

**Definition 2.2.6** Choose \( T \in \mathcal{A}'(\mathbb{R}^n; E) \) whose support is a compact set \( K \). Then we define the **Fourier-Borel transform** \( \hat{T}(\xi) \) of \( T \) by the formula
\[ \hat{T}(\xi) = T_x(e^{-i < x, \xi >}). \]

Then we have the following.

**Proposition 2.2.7** For \( T \in \mathcal{A}'(\mathbb{R}^n; E) \) whose support is a compact set \( K \) in \( \mathbb{R}^n \), \( \hat{T}(\xi) \in \tilde{\mathcal{A}}(\mathbb{R}^n; E) \) and \( \hat{T}(\xi) \) can be extended to the extended complex space \( \mathcal{C}^n \) as a function in \( \tilde{\mathcal{O}}(\mathcal{C}^n; E) \) given by the formula
\[ \hat{T}(\zeta) = T_x(e^{-i < x, \zeta >}). \]

**Proof** It is trivial that \( \hat{T}(\xi) \) is an \( E \)-valued real analytic function. It is a restriction to \( \mathbb{R}^n \) of an \( E \)-valued entire function \( \hat{T}(\zeta) \).
So we have only to prove that \( \hat{T}(\zeta) \in \tilde{O}(\C^n; E) \). For arbitrary positive numbers \( \varepsilon \) and \( \mu \) and for an arbitrary \( \zeta \in \mathbb{R}^n \times i[-\mu, \mu] = L \), we have an estimate

\[
\sup_{\zeta \in L} e(-\varepsilon|\zeta|) \| T_x(e(-i < x, \zeta >)) \|
\]

\[
= \sup_{\zeta \in L} \| T_x(e(-i < x, \zeta > - \varepsilon|\zeta|)) \|
\]

\[
\leq C \sup_{\zeta \in L} e(I_K(\mu) - \varepsilon|\zeta|) < \infty.
\]

Here \( \| \cdot \| \) denotes an arbitrary seminorm defining the topology of \( E \) and we put

\[ I_K(\mu) = \sup \{ \langle x, \mu \rangle; x \in K \}, (\zeta = \xi + i\mu). \]

This completes the proof. \( \text{Q.E.D.} \)

**Corollary.** Let \( T \) be as in Proposition 2.2.7. Then we have the formula \( \mathcal{F}T = \hat{T} \) considering \( T \) as an element in \( \mathcal{A}(\tilde{\mathbb{R}}^n) \).

**Proof** For any \( \varphi \in \mathcal{A}(\tilde{\mathbb{R}}^n) \), we have

\[
\mathcal{F}T(\varphi) = T_x(\int e(-i < x, \xi >)\varphi(\xi)d\xi)
\]

\[
= \int T_x(e(-i < x, \xi >))\varphi(\xi)d\xi = \hat{T}(\varphi).
\]

This completes the proof. \( \text{Q.E.D.} \)

**Theorem 2.2.8 (the Paley-Wiener Theorem)** For an \( E \)-valued Fourier hyperfunction \( T \), the following are equivalent:

1. The support of \( T \) is a compact set in \( \mathbb{R}^n \) whose convex balanced hull is the set \( K \).
The Fourier-Borel transform $\hat{T}(\xi)$ of $T$ can be extended to the extended complex space $\mathbb{C}^n$ as a function $\hat{T}(\zeta)$ in $\tilde{O}(\mathbb{C}^n; E)$ such that, for any positive $\varepsilon$, there exists a constant $C$ such that, for all $\zeta = \xi + i\mu$, the estimate

$$\|\hat{T}(\zeta)\| \leq Ce(I_K(\mu) + \varepsilon|\zeta|)$$

holds. Here $\| \cdot \|$ means an arbitrary seminorm defining the topology of $E$ and we put

$$I_K(\mu) = \sup\{< x, \mu >; x \in K\}.$$

Proof The fact that (1) implies (2) is proved in Proposition 2.2.7.

Now we prove that (2) implies (1). Since $K$ is a closed convex set, we can represent $K = \bigcap H_{\xi, a}$, where we put $H_{\xi, a} = \{x; < x - a, \xi > \geq 0\}$. Then we have the inequality

$$\chi_{H_{\xi, a}}(\eta - a) = \sup\{< x - a, \eta - a > + \varepsilon|x - a|; x \in H_{\xi, a}\}$$

$$\geq I_{K-a}(\eta - a) \geq I_K(\eta) + \text{const}$$

for $\eta - a \in \Gamma' \subset \Gamma^o = (H_{\xi, a} - a)^o$. Thus, $\hat{T}(\zeta)$ satisfies the condition of Theorem 2.2.2 in the case of $\Gamma = H_{\xi, a}$. Hence $T$ belongs to

$$\bigcap \mathcal{A}'(H_{\xi, a}; E) = \mathcal{A}'(\bigcap H_{\xi, a}; E) = \mathcal{A}'(K; E).$$

This completes the proof. Q.E.D.

Remark. In the cases of partial Fourier hyperfunctions and vector valued partial Fourier hyperfunctions, we define the partial Fourier transformation of them, considering the spaces $\mathcal{A}'_b(K \times \mathbb{R}^{m_2})$ and $\mathcal{A}'_b(K \times \mathbb{R}^{m_2}, E)$ as the special case of the space of the space $\mathcal{A}'(\mathbb{R}^m; E)$ for a compact set $K$ in $\mathbb{R}^m$. Thus we can apply the results in this section to them.