Chapter 10

Vector valued partial Fourier hyperfunctions

In this chapter, we study the realization of vector valued partial Fourier hyperfunctions as boundary values of vector valued partially slowly increasing holomorphic functions. In order to do so, we prove the Dolbeault-Grothendieck resolution of $^E\mathcal{O}^p$, the Oka-Cartan-Kawai Theorem B, Malgrange’s Theorem, Serre’s duality theorem, Martineau-Harvey’s Theorem and Sato’s Theorem. In this chapter, we assume $|n| \geq 1$. $\mathcal{N}$ is the set of all natural numbers. In this chapter, we consier the Cauchy-Riemann operator $\bar{\partial}$ in the space $\mathcal{E}^p(\Omega)$ and $\mathcal{E}^s(\Omega; E)$.

Thus, in order to study the theory of vector valued partial Fourier hyperfunctions, we use only the method of the classical analysis.

10.1 The Dolbeault-Grothendieck resolution of $^E\mathcal{O}^p$

In this section, we prove a soft resolution of the sheaf $^E\mathcal{O}^p$. In this chapter, $E$ denotes a Fréchet space. $\mathcal{T} = \mathcal{T}_E$ denotes the family of all continuous seminorms which define the topology of $E$.

For a 2-tuple $n = (n_1, n_2)$ of natural numbers with $|n| = n_1 + n_2 \neq 0$, we denote by $C^{n_1, n_2}$ the product space $C^{n_1} \times C^{n_2}$ and by $R^{n_1, n_2}$ the product space $R^{n_1} \times R^{n_2}$. We also denote by $C^{[n]}$ the product space $C^{[n]} = C^{n_1} \times C^{n_2}$. We denote $z = (z', z'') \in C^{[n]}$ with $z' = (z_1, \cdots, z_{n_1})$ and $z'' = (z_{n_1+1}, \cdots, z_{|n|})$. In this chapter, we assume $|n| \geq 1$.

At first, we define the sheaves $^E\mathcal{O}^p$ and $^E\mathcal{E}^s$.

Definition 10.1.1(The sheaf $^E\mathcal{O}^p$ of germs of $E$-valued partially slowly increasing holomorphic functions over $C^{\phi, n}$) We define the sheaf $^E\mathcal{O}^p$ to be the sheafification of the presheaf $\{\mathcal{O}^p(\Omega; E); \Omega$ is an open set in $C^{\phi, n}\}$. Here, for an open set $\Omega$ in $C^{\phi, n}$, the section module $\mathcal{O}^p(\Omega; E)$ is defined as follows:
\[ O^\flat(\Omega; E) = \{ f \in O(\Omega \cap C^{[n]}; E); \text{ for any positive } \varepsilon \text{ and any compact set } K \text{ in } \Omega \text{ and any } q \in T, \sup \{ q(f(z))e(-\varepsilon|z|); z \in K \cap C^{[n]} \} < \infty \}. \]

**Definition 10.1.2** (The sheaf \( E^\flat \) of germs of \( E \)-valued partially slowly increasing \( C^\infty \)-functions) We define the sheaf \( E^\flat \) to be the sheafification of the presheaf \{\( \mathcal{E}^\flat(\Omega; E); \Omega \text{ is an open set in } C^{\flat, n} \}). \) Here, for an open set \( \Omega \) in \( C^{\flat, n} \), the section module \( \mathcal{E}^\flat(\Omega; E) \) is defined as follows:

\[ \mathcal{E}^\flat(\Omega; E) = \{ f \in \mathcal{E}(\Omega \cap C^{[n]}; E); \text{ for any positive } \varepsilon \text{ and any compact set } K \text{ in } \Omega \text{ and any } \alpha \in \mathbb{N}^{2[n]} \text{ and any } q \in T, \sup \{ q(f^{(\alpha)}(z))e(-\varepsilon|z|); z \in K \cap C^{[n]} \} < \infty \}. \]

Then the sheaf \( E^\flat \) is a soft sheaf.

Here we have the following.

**Theorem 10.1.3** (The Dolbeault-Grothendieck resolution of \( E^\otimes^\flat, p \)) The sequence of sheaves over \( C^{\flat, n} \)

\[ 0 \longrightarrow E^\otimes^\flat, p \longrightarrow E^\otimes^\flat, p, 0 \longrightarrow \overline{\partial} \frac{E^\otimes^\flat, p, 1}{\overline{\partial}} \longrightarrow \cdots \longrightarrow 0 \]

is exact. Here we assume \( p \geq 0 \).

**Proof** We prove this in the similar way to that of Ito [14], Theorem 3.1, p.989. Q.E.D.

**Corollary** For an open set \( \Omega \) in \( C^{\flat, n} \), we have the isomorphisms:

\[ H^q(\Omega, E^\otimes^\flat, p) \cong \left\{ f \in E^\otimes^\flat, p, q(\Omega; E); \overline{\partial}f = 0 \right\} / \left\{ \overline{\partial}g; g \in E^\otimes^\flat, p, q-1(\Omega; E) \right\} \]

for \( p \geq 0 \) and \( q \geq 1 \).

**Proof** We prove this by virtue of Theorem 10.1.3 and Komatsu [39], Theorem II.2.9 and Theorem II.2.19. Q.E.D.
10.2 The Oka-Cartan-Kawai Theorem B

In this section, we prove the Oka-Cartan-Kawai Theorem B for the sheaf $E^b$.

**Theorem 10.2.1 (The Oka-Cartan-Kawai Theorem B)** For any $O^b$-pseudoconvex open set $\Omega$ in $C^{b,n}$, we have the equalities

$$H^q(\Omega, E^b, p) = 0$$

for $p \geq 0$ and $q \geq 1$.

**Proof** Since we have the equalities

$$H^q(\Omega, O^b, p) = 0, \ (p \geq 0, \ q \geq 1)$$

by the Oka-Cartan-Kawai Theorem B for $O^b$, the following complex

$$E^{b, p, 0}(\Omega) \xrightarrow{\overline{\partial}} E^{b, p, 1}(\Omega) \xrightarrow{\overline{\partial}} \cdots$$

$$\xrightarrow{\overline{\partial}} E^{b, p, |n|}(\Omega) \xrightarrow{\overline{partial}} 0$$

is exact by virtue of Theorem 9.2.2. Since $E^{b, p, q}$’s are nuclear Fréchet spaces and $E$ is the Fréchet space, the complex

$$E^{b, p, 0}(\Omega; E) \xrightarrow{\overline{\partial}} E^{b, p, 1}(\Omega; E) \xrightarrow{\overline{\partial}} \cdots$$

$$\xrightarrow{\overline{\partial}} E^{b, p, |n|}(\Omega; E) \xrightarrow{\overline{partial}} 0$$

is also exact by virtue of the isomorphisms

$$E^{b, p, q}(\Omega; E) \cong E^{b, p, q}(\Omega) \hat{\otimes} E, \ (p \geq 0, \ q \geq 0)$$

and Ion and Kawai [9], Theorem 1.10. Hence we obtain the equalities

$$H^q(\Omega, E^b, p) = 0$$

for $p \geq 0$ and $q \geq 1$.

This completes the proof. Q.E.D.

**Corollary** Let $\Omega$ be an $O^b$-pseudoconvex open set in $C^{b,n}$. Then the equation $\overline{\partial}u = f$ has a solution $u \in E^{b, p, q}(\Omega; E)$ for every $f \in E^{b, p, q+1}(\Omega; E)$ such that $\overline{\partial}f = 0$ holds. Here $p$ and $q$ are two natural numbers.

**Proof** We prove this by virtue of Theorem 10.2.1 and Corollary to Theorem 10.1.3. Q.E.D.
10.3 Malgrange’s Theorem

In this section, we prove the Malgrange’s Theorem for the sheaf $\mathcal{E}^\flat$.

**Theorem 10.3.1** Let $\Omega$ be an open set in $\mathcal{C}^\flat$. Then we have the equality

$$H^{[n]}(\Omega, \mathcal{E}^\flat) = 0.$$  

**Proof** By virtue of Theorems 9.2.2 and 9.3.1, we have the exact sequence

$$\mathcal{E}^\flat, 0, [n]|-1(\Omega) \to \mathcal{E}^\flat, 0, [n](\Omega) \to 0.$$  

Thus, by Tréves [59], Proposition 43.9, we have the exact sequence

$$\mathcal{E}^\flat, 0, [n]|-1(\Omega) \otimes E \to \mathcal{E}^\flat, 0, [n](\Omega) \otimes E \to 0.$$  

Namely we have the exact sequence

$$\mathcal{E}^\flat, 0, [n]|-1(\Omega; E) \to \mathcal{E}^\flat, 0, [n](\Omega; E) \to 0.$$  

Hence we obtain the conclusion. Q.E.D.

**Corollary** Flabby dim $\mathcal{E}^\flat \leq |n|$ holds.

10.4 Serre’s duality theorem

In this section, we prove the Serre’s duality theorem.

**Theorem 10.4.1** Let $\Omega$ be an open set in $\mathcal{C}^\flat$. such that dim $H^p(\Omega, \mathcal{E}^\flat) < \infty$ holds for $p \geq 1$. Then we have the isomorphisms

$$H^p(\Omega, \mathcal{E}^\flat) \cong L(H^{[n]-p}(\Omega, \mathcal{O}^\flat), E)$$

for $0 \leq p \leq |n|$.

**Proof** By the similar method to Junker [29], Lemma [3.5], we obtain the isomorphism

$$H^p(\Omega, \mathcal{E}^\flat) \cong H^p(\Omega, \mathcal{O}^\flat) \otimes \pi E.$$  

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Then, by Theorem 10.4.1, we have the following isomorphisms

\[ H^p(\Omega, E^\mathcal{O}) \cong H^p(\Omega, \mathcal{O}) \hat{\otimes}_\mathcal{E} E \]
\[ \cong [H^{|n|+p}(\Omega, \mathcal{O})]^{\prime} \hat{\otimes} \pi E \cong L(H^{|n|-p}(\Omega, \mathcal{O}_b), E). \]

This completes the proof of this theorem. Q.E.D.

10.5 Martineau-Harvey’s Theorem

In this section, we prove the Martineau-Harvey’s Theorem.

**Theorem 10.5.1** Let \( K \) be a compact set in \( \mathcal{C}^d, n \) such that it has an \( \mathcal{O}^d \)-pseudoconvex open neighborhood \( \Omega \) and it satisfies conditions \( H^p(K, \mathcal{O}_b) = 0 \) for \( p \geq 1 \). Then we have the following (1) \sim (3):

1. \( H^p(K, E^\mathcal{O}) = 0 \), holds for \( p \neq |n| \).

2. In the case \(|n| \geq 2\), we have the topological isomorphisms
   \[ H^{|n|}(\Omega, E^\mathcal{O}) \cong H^{|n|-1}(\Omega \setminus K, E^\mathcal{O}) \cong L(\mathcal{O}_b(K), E). \]

3. In the case \(|n| = 1\), we have the topological isomorphisms
   \[ H^{|n|}(\Omega, E^\mathcal{O}) \cong \frac{\mathcal{O}^d(\Omega \setminus K; E)}{\mathcal{O}^d(\Omega; E)} \cong L(\mathcal{O}_b(K), E). \]

**Proof** (A) At first we prove (1). We assume that \( \Omega \) is an \( \mathcal{O}^d \)-pseudoconvex open neighborhood of \( K \). Then, in the long exact sequence of cohomology groups

\[
0 \rightarrow H^0_K(\Omega, E^\mathcal{O}) \rightarrow H^0(\Omega, E^\mathcal{O}) \rightarrow H^0(\Omega \setminus K, E^\mathcal{O}) \\
\rightarrow H^{|n|}_K(\Omega, E^\mathcal{O}) \rightarrow H^{|n|}(\Omega, E^\mathcal{O}) \rightarrow H^{|n|}(\Omega \setminus K, E^\mathcal{O}) \\
\rightarrow \ldots \\
\rightarrow H^{|n|}_K(\Omega, E^\mathcal{O}) \rightarrow H^{|n|}(\Omega, E^\mathcal{O}) \rightarrow H^{|n|}(\Omega \setminus K, E^\mathcal{O}) \\
\rightarrow \ldots,
\]
we have the equalities $H^p(\Omega, \mathcal{E}\mathcal{O}^\flat) = 0$ for $p \geq 1$. Further we have the equality $H^0_K(\Omega, \mathcal{E}\mathcal{O}^\flat) = 0$ by the unique continuation theorem. As for these results, we refer to Komatsu [39], Theorem II.3.2. Hence we have the topological isomorphisms

$$H^1_K(\Omega, \mathcal{E}\mathcal{O}^\flat) \cong \mathcal{O}^h(\Omega \setminus K; \mathcal{E}),$$

$$H^p_K(\Omega, \mathcal{E}\mathcal{O}^\flat) \cong H^{p-1}(\Omega \setminus K, \mathcal{E}\mathcal{O}^\flat), \ (p \geq 2).$$

But, by the similar way to that of Junker [29], Lemma 3.5, we have the topological isomorphisms

$$H^p(V, \mathcal{E}\mathcal{O}^\flat) \cong H^p(V, \mathcal{O}^\flat) \otimes \mathcal{E}, \ (0 \leq p \leq |n|)$$

for any open set $V$ in $C^{\phi, n}$. Therefore, by Theorem 9.5.1, we have the topological isomorphisms

$$H^p_K(\Omega, \mathcal{E}\mathcal{O}^\flat) \cong H^p_K(\Omega, \mathcal{O}^\flat) \otimes \pi \mathcal{E} \cong H^p(\Omega, \mathcal{O}^\flat) \otimes \pi \mathcal{E} = 0.$$

for $p \neq |n|$. This prove (1)

(B) In the case $|n| \geq 2$, we prove (2). We have the isomorphisms

$$H^{|n|}_K(\Omega, \mathcal{E}\mathcal{O}^\flat) \cong H^{|n|}_K(\Omega \setminus K, \mathcal{E}\mathcal{O}^\flat) \cong H^{|n|}_K(\Omega \setminus K, \mathcal{O}^\flat) \otimes \pi \mathcal{E} \cong H^{|n|}_K(\Omega, \mathcal{O}^\flat) \otimes \pi \mathcal{E} \cong \mathcal{O}_s(K) \otimes \mathcal{E} \cong L(\mathcal{O}_s(K), \mathcal{E}).$$

This proves (2)

(C) In the case $|n| = 1$, we prove (3). By the similar way, we have the topological isomorphisms

$$H^1_K(\Omega, \mathcal{E}\mathcal{O}^\flat) \cong \mathcal{O}^h(\Omega \setminus K; \mathcal{E}) \cong H^1_K(\Omega, \mathcal{O}^\flat) \otimes \mathcal{E} \cong \mathcal{O}_s(K) \otimes \mathcal{E} \cong L(\mathcal{O}_s(K), \mathcal{E}).$$

This proves (3).

This completes the proof of this theorem. Q.E.D.
10.6 Sato’s Theorem

In this section, we prove the pure-codimensionality of $\mathbf{R}^\flat, n$ with respect to the sheaf $E\mathcal{O}^\flat$. Then we realize $E$-valued partial Fourier hyperfunctions as “boundary values” of $E$-valued partially slowly increasing holomorphic functions or as relative cohomology classes of $E$-valued partially slowly increasing holomorphic functions.

**Theorem 10.6.1 (Sato’s Theorem)** Let $\Omega$ be an open set in $\mathbf{R}^\flat, n$ and $V$ an open set in $\mathbf{C}^\flat, n$ which contains $\Omega$ as its closed subset. Then we have the following (1) ~ (3):

1. The relative cohomology groups $H^p_\Omega(V, E\mathcal{O}^\flat)$ are zero for $p \neq |n|$.
2. The presheaf over $\mathbf{R}^\flat, n$

$$\Omega \longrightarrow H^{|n|}_\Omega(V, E\mathcal{O}^\flat)$$

is a flabby sheaf.

3. This sheaf (2) is isomorphic to the sheaf $E\mathcal{B}^\flat$ of $E$-valued partial Fourier hyperfunctions realized by the duality method in Definition 11.2.3 of Part I of this book.

**Proof** (A) By the excision theorem, we assume that $V$ is an $\mathcal{O}^\flat$-pseudoconvex open set in $\mathbf{C}^\flat, n$. Consider the following exact sequence of relative cohomology groups

$$
0 \longrightarrow H^0_{\partial\Omega}(V, E\mathcal{O}^\flat) \longrightarrow H^0_\Omega(V, E\mathcal{O}^\flat) \longrightarrow H^0_{\Omega^a}(V, E\mathcal{O}^\flat) \longrightarrow H^1_{\partial\Omega}(V, E\mathcal{O}^\flat) \longrightarrow \cdots \longrightarrow H^{|n|}_\Omega(V, E\mathcal{O}^\flat) \longrightarrow H^{|n|+1}_\Omega(V, E\mathcal{O}^\flat) \longrightarrow \cdots.
$$

By virtue of Theorems 10.5.1, we have the equalities

$$H^p_{\partial\Omega}(V, E\mathcal{O}^\flat) = H^p_{\Omega^a}(V, E\mathcal{O}^\flat) = 0$$

for $p \neq |n|$. Therefore, we have the equalities

$$H^p_\Omega(V, E\mathcal{O}^\flat) = 0$$
for $p \neq |n| - 1$, $|n|$. On the other hand, by virtue of Theorems 10.5.1, we have the exact sequence

$$0 \longrightarrow H^{[n]-1}_\Omega(V, E\mathcal{O}^\flat) \longrightarrow L(A_0(\partial\Omega), E) \longrightarrow L(A_0(\Omega^n), E).$$

Since $j$ is injective, we have

$$H^{[n]-1}_\Omega(V, E\mathcal{O}^\flat) = 0.$$

This proves (1).

(B) By virtue (1) and Corollary to Theorem 10.3.1 and Komatsu [39], Theorem II.3.24, we have the conclusion. This prove (2)

(C) By the proof of (1), we have the exact sequence

$$0 \longrightarrow H^{[n]}_{\partial\Omega}(V, E\mathcal{O}^\flat) \longrightarrow H^{[n]}_{\Omega^n}(V, E\mathcal{O}^\flat) \longrightarrow H^{[n]}_\Omega(V, E\mathcal{O}^\flat) \longrightarrow 0.$$

Since we have the topological isomorphisms

$$H^{[n]}_{\partial\Omega}(V, E\mathcal{O}^\flat) \cong L(A_0(\partial\Omega), E),$$

$$H^{[n]}_{\Omega^n}(V, E\mathcal{O}^\flat) \cong L(A_0(\Omega^n), E),$$

by Martineau-Hervey’s Theorem, we have the algebraic isomorphisms

$$H^{[n]}_\Omega(V, E\mathcal{O}^\flat) \cong \frac{L(A_0(\Omega^n), E)}{L(A_0(\partial\Omega), E)} \cong \mathcal{B}(\Omega; E).$$

Thus the sheaf $\Omega \longrightarrow H^{[n]}_\Omega(V, E\mathcal{O}^\flat)$ is isomorphic to the sheaf $E\mathcal{B}^\flat$ of $E$-valued partial Fourier hyperfunctions over $R^{\flat, n}$. This proves (3).

This completes the proof of this theorem. Q.E.D.

By virtue of Theorem 10.6.1, we realize the $E$-valued partial Fourier hyperfunctions as boundary values of $E$-valued partially slowly increasing holomorphic functions or as the relative cohomology classes with coefficients in $E\mathcal{O}^\flat$. Thus we prove that two realizations of the $E$-valued partial Fourier hyperfunctions as the classes of $E$-valued partial Fourier analytic linear mappings and as the boundary values of $E$-valued partially slowly increasing holomorphic functions are constructed independently and they are equivalent.

Here, we have the following.
**Theorem 10.6.2**  In the notation such as that of Theorem 9.6.2, we have the algebraic isomorphisms

\[ H^{[n]}_\Omega(V, E^\mathcal{O} \vert) \cong H^{[n]}(V, \mathcal{O}', E^\mathcal{O} \vert) \cong \frac{\mathcal{O}^\flat(\bigcap_j V_j; E)}{\sum_{j=1}^{[n]} \mathcal{O}^\flat(\bigcap_{i \neq j} V_i; E)}. \]

At last, we realize partial Fourier analytic linear mappings with certain compact carrier as relative cohomology classes with coefficients in \( E^\mathcal{O} \vert \).

Let \( K \) be a compact set in \( \mathcal{C}^\flat, n \) of the form \( K = K_1 \times \cdots \times K_{[n]} \) with the compact sets \( K_1, K_2, \cdots, K_{[n]} \) in \( \mathcal{C} \) and \( K_{n_1+1}, K_{n_1+2}, \cdots, K_{[n]} \) in \( \tilde{\mathcal{C}} \). Assume that \( K \) admits a fundamental system of \( \mathcal{O}^\flat \)-pseudoconvex open neighborhoods. Then we have equalities

\[ H^p(K, \mathcal{O}_x) = 0 \]

for \( p \geq 0 \).

By virtue of Martineau-Harvey’s Theorem, we have the topological isomorphism

\[ \mathcal{O}_x'(K; E) \cong H^{[n]}_K(C^\flat, n, \mathcal{O} \vert). \]

Further assume that there exists an \( \mathcal{O}^\flat \)-pseudoconvex open neighborhood \( \Omega \) such that

\[ \Omega_j = \Omega \setminus \{ z \in C^\flat, n; z_j \in K_j \cap C^{[n]}_{j} \} \]

is also an \( \mathcal{O}^\flat \)-pseudoconvex open set for \( j = 1, 2, \cdots, [n] \). Put \( \Omega_0 = \Omega \). Then \( \mathcal{V} = \{ \Omega_0, \Omega_1, \cdots, \Omega_{[n]} \} \) and \( \mathcal{V}' = \{ \Omega_1, \Omega_2, \cdots, \Omega_{[n]} \} \) form acyclic coverings of \( \Omega \) and \( \Omega \setminus K \). Set

\[ \Omega^*_K = \bigcap_{j=1}^{[n]} \Omega_j, \quad \Omega^j = \bigcap_{i \neq j} \Omega_i, \quad (1 \leq j \leq [n]). \]

Let \( \sum_j \mathcal{O}^\flat(\Omega^j; E) \) be the image in \( \mathcal{O}^\flat(\Omega^*_K; E) \) of \( \prod_{j=1}^{[n]} \mathcal{O}^\flat(\Omega^j; E) \) by the mapping

\[ (f_j)_{j=1}^{[n]} \rightarrow \sum_{j=1}^{[n]} (-1)^{j+1} f'_j \]
where \( f'_j \) denotes the restriction of \( f_j \) to \( \Omega \sharp K \).

Then, by the similar way to that of Theorem 10.6.2, we have the following.

**Theorem 10.6.3** We use the notation in the above. Then we have the topological isomorphisms

\[
\mathcal{O}^p(K; E) \cong H^{[n]}_K(C^\omega, \, E^{\mathcal{O}^p}) \cong H^{[n]}(\mathcal{V}, \, \mathcal{V}', \, E^{\mathcal{O}^p}) \cong \frac{\mathcal{O}^p(\Omega \sharp K; E)}{\sum_j \mathcal{O}^p(\Omega_j; E)}.
\]

The canonical mapping

\[
b : \mathcal{O}^p(\Omega \sharp K; E) \to \mathcal{O}'_p(K; E)
\]

is surjective and its kernel is equal to

\[b^{-1}(0) = \sum_j \mathcal{O}^p(\Omega_j; E), \,(|n| > 1),\]

or

\[b^{-1}(0) = \mathcal{O}^p(\Omega; E), \,(|n| = 1).\]

Then we have the following.

**Theorem 10.6.4** We use the notation in the above. Then we have the following (1) and (2):

1. Choose \( f \in \mathcal{O}^p(\Omega \sharp K; E) \) and \( g \in \mathcal{O}_b(K) \). We choose \( \omega = \omega_1 \times \cdots \times \omega_{|n|} \) is an open set in \( \Omega \). Here \( \omega_j \) is the open neighborhood of \( K_j \) in \( C \) or \( \bar{C} \), \((1 \leq j \leq |n|)\) and \( \bar{\omega} \) is an open neighborhood of \( \omega \) in \( \Omega \) such that \( g \in \mathcal{O}_b(\bar{\omega}) \) holds. Let \( \Gamma_j \) be a regular contour in \( \omega_j \cap C \) enclosing once \( K_j \cap C \) and oriented in the usual way, \((1 \leq j \leq |n|)\). Then we have the equality

\[
b(f)(g) = (-1)^{|n|} \int_{\Gamma_1} \cdots \int_{\Gamma_{|n|}} f(z)g(z)dz_1 \cdots dz_{|n|}.
\]

Then this linear mapping \( b \) defines the canonical mapping in the Theorem 10.6.3.
(2) Choose \( u \in \mathcal{O}_h'(K; E) \) and put
\[
\tilde{u}(z) = \frac{1}{(2\pi i)^n} u_\xi \left( \frac{1}{\xi - z} \exp\left( -(\xi'' - z'')^2 \right) \right).
\]

Here we write
\[
\frac{1}{\xi - z} = \prod_{j=1}^{[n]} \frac{1}{\xi_j - z_j},
\]
\[
(\xi'' - z'')^2 = \sum_{j=n_1+1}^{[n]} (\xi_j - z_j)^2.
\]

Then we have \( \tilde{u} \in \mathcal{O}_h^b(\Omega^w_h K; E) \) and \( b(\tilde{u}) = u \).

**Proof** (A) At first, we prove (1). Using the notation in the above, we put
\[
b_1(f)(g) = (-1)^{|n|} \int_{\Gamma_1} \cdots \int_{\Gamma_{[n]}} f(z) g(z) dz_1 \cdots dz_{[n]}.
\]
This integral does not depend on the chosen contours. Hence this defines a linear mapping
\[
b_1 : \mathcal{O}_h^b(\Omega^w_h K; E) \to \mathcal{O}_h'(K; E).
\]
Then the kernel of \( b_1 \) is equal to
\[
b_1^{-1}(0) = \sum_j \mathcal{O}_h^b(\Omega^j; E), \ (|n| > 1)
\]
and
\[
b_1^{-1}(0) = \mathcal{O}_h^b(\Omega; E), \ (|n| = 1).
\]

Thus this linear mapping \( b_1 \) is equal to the canonical mapping \( b \) in Theorem 10.6.3. Hence we have the equality
\[
b_1(f) = b(f)
\]
This proves (1).

(B) Next we prove (2). It suffices to prove that, if \( u \in \mathcal{O}_h'(K; E) \) holds, we have the equality
\[
b(\tilde{u}) = u.
\]
But, for $g \in \mathcal{O}(K)$, we have the equality

$$b(\tilde{u})(g) = \frac{(-1)^{|n|}}{(2\pi i)^{|n|}} \int_{\Gamma_1} \cdots \int_{\Gamma_{|n|}} u_{\xi} \left( \frac{1}{\xi - z} \exp(-((\xi'' - z'')^2)) \right) g(z) dz$$

$$= u_{\xi} \left( \frac{1}{(2\pi i)^{|n|}} \int_{\Gamma_1} \cdots \int_{\Gamma_{|n|}} g(z) \left( \frac{1}{z - \xi} \exp(-((z'' - \xi'')^2)dz) \right) \right)$$

$$= u(g).$$

Thus we have the equality $b(\tilde{u}) = u$. This proves (2).

This completes the proof of this theorem. Q.E.D.