Chapter 4

Vector valued Fourier hyperfunctions

In this chapter, we study the realization of vector valued Fourier hyperfunctions as boundary values of vector valued slowly increasing holomorphic functions. In order to do so, we prove the the Dolbeault-Grothendieck resolution of $^E\mathcal{O}$, the Oka-Cartan-Kawai Theorem B, Malgrange’s Theorem, Serre’s duality theorem, Martineau-Harvey’s Theorem and Sato’s Theorem.

In this chapter, we assume that $n \geq 1$ holds and $\mathbb{N}$ is the set of all natural numbers.

In this chapter, we consider the Cauchy-Riemann operator $\partial$ in the spaces $^E\mathcal{E}(\Omega)$ and $^E\mathcal{E}(\Omega; E)$. Thus, in order to study the theory of vector valued Fourier hyperfunctions, we use only the method of the classical analysis.

4.1 The Dolbeault-Grothendieck resolution of the sheaf $^E\mathcal{O}$

In this section, we prove the soft resolution of the sheaf $^E\mathcal{O}$. Here, $E$ denotes a Frechét space. $\mathcal{T} = \mathcal{T}_E$ denotes the family of all seminorms which define the topology of $E$.

We assume $n \geq 1$. We denote by $\tilde{R}^n$ the radial compactification of $R^n$ in the sense of Kawai. We denote by $\tilde{C}^n$ the space $\tilde{R}^n \times \sqrt{-1}R^n$ endowed with the direct product topology.

At first, we define the sheaves $^E\mathcal{O}$ and $^E\mathcal{E}$. As for these facts, we refer to Ito [14] and Junker [28].

Definition 4.1.1(The sheaf $^E\mathcal{O}$ of germs of $E$-valued slowly increasing holomorphic functions over $\tilde{C}^n$). We define the sheaf
$E\tilde{\mathcal{O}}$ to be the sheafification of the presheaf $\{\tilde{\mathcal{O}}(\Omega; E); \Omega \text{ is an open set in } \mathbb{C}^n\}$. Here, for an open set $\Omega$ in $\mathbb{C}^n$, the section module $\tilde{\mathcal{O}}(\Omega; E)$ is defined as follows:

$$\tilde{\mathcal{O}}(\Omega; E) = \{f \in \mathcal{O}(\Omega \cap \mathbb{C}^n; E); \text{for any positive } \varepsilon \text{ and any compact set } K \text{ in } \Omega \text{ and any } q \in T, \sup\{q(f(z))e(-\varepsilon|z|); z \in K \cap \mathbb{C}^n\} < \infty\}.$$ 

Here we denote by $\mathcal{O}(\Omega \cap \mathbb{C}^n; E)$ the space of all $E$-valued holomorphic functions on the open set $\Omega \cap \mathbb{C}^n$.

We define that this sheaf $E\tilde{\mathcal{O}}$ is the sheaf of germs of $E$-valued slowly increasing holomorphic functions.

**Definition 4.1.2 (The sheaf $E\tilde{\mathcal{E}}$ of germs of $E$-valued slowly increasing $C^\infty$-functions).** We define the sheaf $E\tilde{\mathcal{E}}$ to be the sheafification of the presheaf $\{\tilde{\mathcal{E}}(\Omega; E); \Omega \text{ is an open set in } \mathbb{C}^n\}$. For an open set $\Omega$ in $\mathbb{C}^n$, the section module $\tilde{\mathcal{E}}(\Omega; E)$ is defined as follows:

$$\tilde{\mathcal{E}}(\Omega; E) = \{f \in \mathcal{E}(\Omega \cap \mathbb{C}^n; E); \text{for any positive } \varepsilon \text{ and any compact set } K \text{ in } \Omega \text{ and any } \alpha \in \mathbb{N}^{2n}. \text{ and any } q \in T, \sup\{q(f^{(\alpha)}(z))e(-\varepsilon|z|); z \in K \cap \mathbb{C}^n\} < \infty\}.$$ 

Here $\mathbb{N}$ is the set of all natural numbers and $\mathcal{E}(\Omega \cap \mathbb{C}^n; E)$ is the space of all $E$-valued $C^\infty$-functions on the open set $\Omega \cap \mathbb{C}^n$.

Then the sheaf $E\tilde{\mathcal{E}}$ is a soft sheaf.

Thus, we have the following.

**Theorem 4.1.3 (The Dolbeault-Grothendieck resolution of $E\tilde{\mathcal{O}}^p$).** Let $E$ be a Fréchet space. Then, for $p \in \mathbb{N}$, the sequence of sheaves over $\mathbb{C}^n$

\[
\begin{array}{ccccccc}
0 & \longrightarrow & E\tilde{\mathcal{O}}^p & \longrightarrow & E\tilde{\mathcal{E}}^{p.0} & \longrightarrow & E\tilde{\mathcal{E}}^{p.1} \\
& & \big\uparrow \partial & & \big\uparrow \partial & & \vdots \\
& & E\tilde{\mathcal{E}}^{p.n} & \longrightarrow & 0
\end{array}
\]
is exact. Here we denote by $F^{p,q}$ the sheaf of germs of differential forms of type $(p, q)$ with coefficients in a sheaf $F$ and put $F^p = F^{p,0}$. Here we assume $p, q \in \mathbb{N}$.

**Proof** We refer to Ito [14], Theorem 3.1, p. 989. Q.E.D.

**Corollary** For an open set $\Omega$ in $\tilde{C}^n$, we have the following isomorphisms:

$$H^q(\Omega, \mathcal{E}^p) \cong \left\{ f \in \mathcal{E}^p \Omega : \bar{\partial}f = 0 \right\} \big/ \left\{ \bar{\partial}g : g \in \mathcal{E}^{p-1} \Omega \right\}, \quad (p \geq 0, q \geq 1).$$

**Proof** We prove this by virtue of Theorem 4.1.3 and Komatsu [39], Theorems II.2.19. Q.E.D.

### 4.2 The Oka-Cartan-Kawai Theorem B

In this section, we prove the Oka-Caltan-Kawai Theorem B for the sheaf $\mathcal{E}^p$.

**Theorem 4.2.1 (The Oka-Cartan-Kawai Theorem B)** Let $E$ be a Fréchet space. For any $\tilde{\mathcal{O}}$-pseudoconvex open set $\Omega$ in $\tilde{C}^n$, we have the equalities

$$H^q(\Omega, \mathcal{E}^p) = 0$$

for $p \geq 0$ and $q \geq 1$.

**Proof** We refer to Ito [14], p.992 or Junker [29], p.33. Q.E.D.

**Corollary** Let $E$ be a Fréchet space and $\Omega$ an $\tilde{\mathcal{O}}$-pseudoconvex open set. Then the equation $\bar{\partial}u = f$ has a solution $u \in \mathcal{E}^{p,q+1} \Omega$ for every $f \in \mathcal{E}^{p,q+1} \Omega$ such that $\bar{\partial}f = 0$ holds. Here $p, q$ are two natural numbers.

**Proof** We prove this by virtue of Theorem 4.2.1 and Corollary to Theorem 4.1.3. Q.E.D.
4.3 Malgrange’s Theorem

In this section, we prove the Malgrange’s Theorem. As for the results of this chapter, we refer to Junker [27]. In the following of this chapter, we assume that $E$ is a Fréchet space.

**Theorem 4.3.1** Let $\Omega$ be an open set in $\tilde{\mathbb{C}}^n$. Then we have the equality $H^n(\Omega, E\tilde{O}) = 0$.

**Proof** We refer to Junker [27], p.34. Q.E.D.

**Corollary** Flabby dim $E\tilde{O} \leq n$ holds.

4.4 Serre’s duality theorem

In this section, we prove the Serre’s duality theorem.

**Theorem 4.4.1** Let $\Omega$ be an open set in $\tilde{\mathbb{C}}^n$ such that

$$\dim H^p(\Omega, \tilde{O}) < \infty$$

holds for $p \geq 1$. Then we have the topological isomorphisms

$$H^p(\Omega, E\tilde{O}) \cong L(H_c^{n-p}(\Omega, \mathcal{O}), E)$$

for $0 \leq p \leq n$.

**Proof** By Junker [27], Lemma 3.5, we have the topological isomorphism $H^p(\Omega, E\tilde{O}) \cong H^p(\Omega, \tilde{O}) \hat{\otimes}_\pi E$. Then, by Theorem 3.4.1, we have the following topological isomorphisms

$$H^p(\Omega, E\tilde{O}) \cong H^p(\Omega, \tilde{O}) \hat{\otimes}_\pi E \cong [H_c^{n-p}(\Omega, \mathcal{O})]' \hat{\otimes}_\pi E$$

$$\cong L(H_c^{n-p}(\Omega, \mathcal{O}), E).$$

This completes the proof. Q.E.D.
4.5 Martineau-Harvey’s Theorem

In this section, we prove the Martineau-Harvey’s Theorem.

**Theorem 4.5.1** Let $K$ be a compact set in $\mathbb{C}^n$ such that it has an $\tilde{\Omega}$-pseudoconvex open neighborhood $\Omega$ and satisfies the conditions $H^p(K, \mathcal{O}) = 0$ for $p \geq 1$. Then we have the following (1) $\sim$ (3):

1. We have the equalities
   \[ H^p_K(\Omega, E\tilde{\Omega}) = 0 \]
   for $p \neq n$.
2. In the case $n \geq 2$, we have the topological isomorphisms
   \[ H^n_K(\Omega, E\tilde{\Omega}) \cong H^{n-1}(\Omega\setminus K, E\tilde{\Omega}) \cong L(\mathcal{O}(K), E). \]
3. In the case $n = 1$, we have the topological isomorphisms
   \[ H^1_K(\Omega, E\tilde{\Omega}) \cong \frac{\tilde{\Omega}(\Omega\setminus K; E)}{\tilde{\Omega}(\Omega; E)} \cong L(\mathcal{O}(K), E). \]

**Proof** (A) We assume that $\Omega$ is an $\tilde{\Omega}$-pseudoconvex open neighborhood of $K$. Then, in the long exact sequence of cohomology groups

\[
0 \longrightarrow H^0_K(\Omega, E\tilde{\Omega}) \longrightarrow H^0(\Omega, E\tilde{\Omega}) \longrightarrow H^0(\Omega\setminus K, E\tilde{\Omega}) \longrightarrow H^1_K(\Omega, E\tilde{\Omega}) \longrightarrow H^1(\Omega, E\tilde{\Omega}) \longrightarrow H^1(\Omega\setminus K, E\tilde{\Omega}) \longrightarrow \cdots
\]

we have the equalities $H^p(\Omega, E\tilde{\Omega}) = 0$ for $p \geq 1$. Further we have the equality $H^p_K(\Omega, E\tilde{\Omega}) = 0$ by the unique continuation theorem. As for
these results, we refer to Komatsu [39], Theorem II.3.2. Hence we have the topological isomorphisms:

\[ H^1_K(\Omega, \mathcal{E}\mathcal{O}) \cong \mathcal{O}(\Omega \setminus K; \mathcal{E}) / \mathcal{O}(\Omega; \mathcal{E}) \]

and we have the topological isomorphisms

\[ H^p_K(\Omega, \mathcal{E}\mathcal{O}) \cong H^{p-1}(\Omega \setminus K; \mathcal{E}\mathcal{O}) \]

for \( p \geq 2 \). But we have the topological isomorphisms

\[ H^p(\Omega, \mathcal{E}\mathcal{O}) \cong H^p(\Omega, \mathcal{O}) \hat{\otimes} \mathcal{E} \]

for \( 0 \leq p \leq n \) and for any open set \( \Omega \) in \( \mathcal{C}^n \) by virtue of Junker [17], Lemma 3.5. Therefore we have the topological isomorphisms

\[ H^p_K(\Omega, \mathcal{E}\mathcal{O}) \cong H^p_K(\Omega, \mathcal{O}) \hat{\otimes} \mathcal{E} = 0 \]

for \( p \neq n \) by Theorem 3.5.1. This proves (1).

(B) In the case \( n \geq 2 \), we prove (2). By the similar way as above, we have the topological isomorphisms

\[ H^n_K(\Omega, \mathcal{E}\mathcal{O}) \cong H^{n-1}(\Omega \setminus K, \mathcal{E}\mathcal{O}) \cong H^{n-1}(\Omega \setminus K, \mathcal{O}) \hat{\otimes} \mathcal{E} \]

\[ \cong H^n_K(\Omega, \mathcal{O}) \hat{\otimes} \mathcal{E} \cong \mathcal{O}(K) \hat{\otimes} \mathcal{E} \cong L(\mathcal{O}(K), \mathcal{E}) \]

This proves (2).

(C) In the case \( n = 1 \), we prove (3). By the similar way, we have the topological isomorphisms

\[ H^1_K(\Omega, \mathcal{E}\mathcal{O}) \cong \mathcal{O}(\Omega \setminus K; \mathcal{E}) / \mathcal{O}(\Omega; \mathcal{E}) \cong H^1_K(\Omega, \mathcal{O}) \hat{\otimes} \mathcal{E} \]

\[ \cong \mathcal{O}(K) \hat{\otimes} \mathcal{E} \cong L(\mathcal{O}(K), \mathcal{E}) \]

This proves (3).

This completes the proof of this theorem. Q.E.D.
4.6 Sato’s Theorem

In this section, we prove the pure-codimensionality of $\tilde{\mathbb{R}}^n$ with respect to $E\tilde{\mathcal{O}}$. Then we realize $E$-valued Fourier hyperfunctions as boundary values of $E$-valued slowly increasing holomorphic functions or as relative cohomology classes of $E$-valued slowly increasing holomorphic functions.

**Theorem 4.6.1 (Sato’s Theorem)** Let $\Omega$ be an open set in $\tilde{\mathbb{R}}^n$ and $V$ an open set in $\tilde{\mathbb{C}}^n$ which contains $\Omega$ as its closed subset. Then we have the following (1) $\sim$ (3).

1. We have the equalities
   \[ H^p(V, E\tilde{\mathcal{O}}) = 0 \]
   for $p \neq n$.

2. The presheaf over $\tilde{\mathbb{R}}^n$
   \[ \Omega \rightarrow H^n_{\Omega}(V, E\tilde{\mathcal{O}}) \]
   is a flabby sheaf.

3. This sheaf (2) is isomorphic to the sheaf $E\tilde{\mathcal{B}}$ of $E$-valued Fourier hyperfunctions realized by the duality method in Definition 5.2.3 of the Part I of this book.

**Proof** (A) By the excision theorem, we may assume that $V$ is an $\tilde{\mathcal{O}}$-pseudoconvex open set in $\tilde{\mathbb{C}}^n$. Consider the following exact sequence of relative cohomology groups

\[
0 \rightarrow H^0_{\partial\Omega}(V, E\tilde{\mathcal{O}}) \rightarrow H^0_{\Omega}(V, E\tilde{\mathcal{O}}) \rightarrow H^0_{\Omega}(V, E\tilde{\mathcal{O}}) \rightarrow \cdots
\]

\[
\rightarrow H^1_{\partial\Omega}(V, E\tilde{\mathcal{O}}) \rightarrow \cdots \rightarrow H^{n-1}_{\Omega}(V, E\tilde{\mathcal{O}})
\]

\[
\rightarrow H^n_{\partial\Omega}(V, E\tilde{\mathcal{O}}) \rightarrow H^n_{\Omega}(V, E\tilde{\mathcal{O}}) \rightarrow H^n_{\Omega}(V, E\tilde{\mathcal{O}}) \rightarrow \cdots
\]

\[
\rightarrow H^{n+1}_{\partial\Omega}(V, E\tilde{\mathcal{O}}) \rightarrow \cdots.
\]
By Theorem 3.1.8 and 4.5.1, we have
\[ H^p_{\partial \Omega}(V, E\tilde{\mathcal{O}}) = H^p_{\Omega a}(V, E\tilde{\mathcal{O}}) = 0 \]
for \( p \neq n \). Therefore, we have
\[ H^p_{\Omega}(V, E\tilde{\mathcal{O}}) = 0 \]
for \( p \neq n - 1, n \). On the other hand, by Theorems 3.1.8 and 4.5.1, we have the exact sequence
\[ 0 \rightarrow H^{n-1}_{\Omega}(V, E\tilde{\mathcal{O}}) \rightarrow L(A(\partial\Omega), E) \rightarrow j^* L(A(\Omega^n), E) \rightarrow 0. \]

Since \( j \) is injective, we have \( H^{n-1}_{\Omega}(V, E\tilde{\mathcal{O}}) = 0 \). This proves (1).

(B) By Malgrange’s Theorem, we have \( \text{flabby dim} E\tilde{\mathcal{O}} \leq n \). Thus, by (1) and by the theorem II.3.24 of Komatsu [39], we have the conclusion. This proves (2).

(C) By the proof of (1), we have the exact sequence
\[ 0 \rightarrow H^n_{\partial\Omega}(V, E\tilde{\mathcal{O}}) \rightarrow H^n_{\Omega a}(V, E\tilde{\mathcal{O}}) \rightarrow j^* H^n_{\Omega}(V, E\tilde{\mathcal{O}}) \rightarrow 0. \]

Since we have the topological isomorphisms
\[ H^n_{\partial\Omega}(V, E\tilde{\mathcal{O}}) \cong L(A(\partial\Omega), E), \]
\[ H^n_{\Omega a}(V, E\tilde{\mathcal{O}}) \cong L(A(\Omega^n), E) \]
by Martineau-Harvey’s Theorem, we have the algebraic isomorphism
\[ H^n_{\Omega}(V, E\tilde{\mathcal{O}}) \cong \frac{L(A(\Omega^n), E)}{L(A(\partial\Omega), E)} = \tilde{B}(\Omega; E). \]

Thus the sheaf \( \Omega \rightarrow H^n_{\Omega}(V, E\tilde{\mathcal{O}}) \) is isomorphic to the sheaf \( E\tilde{B} \) of \( E \)-valued Fourier hyperfunctions over \( \mathbb{R}^n \). This proves (3).

This completes the proof. Q.E.D.

By virtue of Theorem 4.6.1, we realize the \( E \)-valued Fourier hyperfunctions as boundary values of \( E \)-valued slowly increasing holomorphic
functions or as the relative cohomology classes with coefficients in $\mathcal{E}\tilde{\mathcal{O}}$.

Thus we prove that two realizations of the $\mathcal{E}$-valued Fourier hyperfunctions as the classes of $\mathcal{E}$-valued Fourier analytic linear mappings and as the boundary values of $\mathcal{E}$-valued slowly increasing holomorphic functions are constructed independently and they are equivalent.

Here we have the following.

**Theorem 4.6.2** We use the notation in Theorem 1.6.2. Then we have the algebraic isomorphisms

$$H^n_\Omega(V, \mathcal{E}\tilde{\mathcal{O}}) \cong H^n(\mathcal{V}, \mathcal{V}', \mathcal{E}\tilde{\mathcal{O}}) \cong \frac{\tilde{\mathcal{O}}(\bigcap_j V_j; E)}{\sum_{j=1}^n \tilde{\mathcal{O}}(\bigcap_{i \neq j} V_i; E)}.$$

At last, we realize Fourier analytic linear mappings with certain compact carrier as relative cohomology classes with coefficients in $\mathcal{E}\tilde{\mathcal{O}}$.

Let $K$ be a compact set in $\mathbb{C}^n$ of the form $K = K_1 \times \cdots \times K_n$ with compact set $K_j$ in $\mathbb{C}$, $(j = 1, 2, \cdots, n)$. Assume that $K$ admits a fundamental system of $\tilde{\mathcal{O}}$-pseudoconvex open neighborhoods. Then we have

$$H^p(K, \mathcal{O}) = 0$$

for $p > 0$. By virtue of Martineau-Harvey’s Theorem, there exists the topological isomorphism

$$\mathcal{O}'(K; E) \cong H^n_K(\mathcal{C}^n, \mathcal{E}\tilde{\mathcal{O}}).$$

Further assume that there exists an $\tilde{\mathcal{O}}$-pseudoconvex open neighborhood $\Omega$ of $K$ such that

$$\Omega_j = \Omega \cap \{z \in \mathbb{C}^n; z_j \notin K_j\}$$

is also an $\tilde{\mathcal{O}}$-pseudoconvex open set for $j = 1, 2, \cdots, n$. Then $\mathcal{V} = \{\Omega_0, \Omega_1, \cdots, \Omega_n\}$ and $\mathcal{V}' = \{\Omega_1, \Omega_2, \cdots, \Omega_n\}$ form the acyclic coverings of $\Omega$ and $\Omega \setminus K$. Here we put $\Omega_0 = \Omega$. Further we put

$$\Omega\#K = \bigcap_{j=1}^n \Omega_j, \quad \Omega_j = \bigcap_{i \neq j} \Omega_i, \quad (1 \leq j \leq n).$$
Let $\sum_j \tilde{\mathcal{O}}(\Omega^j; E)$ be the image in $\tilde{\mathcal{O}}(\Omega#K; E)$ of $\prod_{j=1}^n \tilde{\mathcal{O}}(\Omega^j; E)$ by the mapping

$$(f_j)_{j=1}^n \mapsto \sum_{j=1}^n (-1)^{j+1} f'_j,$$

where $f'_j$ denotes the restriction of Theorem 3.6.2.

Then we have the following.

**Theorem 4.6.3** We use the notation in the above. Then we have the topological isomorphisms

$$O'(K; E) \cong H^0_K(\mathcal{C}^n, E\tilde{\mathcal{O}}) \cong H^0(\mathcal{V}, \mathcal{V}', E\tilde{\mathcal{O}}) \cong \tilde{\mathcal{O}}(\Omega#K; E).$$

The canonical mapping

$$b: \tilde{\mathcal{O}}(\Omega#K; E) \longrightarrow O'(K; E)$$

is surjective and its kernel is equal to

$$b^{-1}(0) = \sum_j \tilde{\mathcal{O}}(\Omega^j; E), \ (n > 1),$$

or

$$b^{-1}(0) = \tilde{\mathcal{O}}(\Omega; E), \ (n = 1).$$

Then we have the following.

**Theorem 4.6.4** We use the notation in the above. Then we have the following (1) and (2):

1. Choose $f \in \tilde{\mathcal{O}}(\Omega#K; E)$ and $g \in \mathcal{O}(K)$. We choose $\omega = \omega_1 \times \cdots \times \omega_n \subset \Omega$ with an open neighborhood $\omega_j$ of $K_j$ in $\mathcal{C}$, $(1 \leq j \leq n)$ and $g \in \mathcal{O}(\tilde{\omega})$, where $\tilde{\omega}$ is an open neighborhood of $\omega$ with $\omega \subset \Omega$. Let $\Gamma_j$ be a regular contour in $\omega_j \cap \mathcal{C}$ enclosing once $K_j \cap \mathcal{C}$ and oriented in the usual way, $(j = 1, 2, \cdots, n)$. Then we have the equality

$$b(f)(g) = (-1)^n \int_{\Gamma_1} \cdots \int_{\Gamma_n} f(z)g(z)dz_1 \cdots dz_n.$$

Then this linear mapping $b$ defines the canonical mapping $b$ in the Theorem 4.6.3.
Choose \( u \in \mathcal{O}'(K; E) \) and put

\[
\tilde{u}(z) = \frac{1}{(2\pi i)^n} u_{\xi}\left( \prod_{j=1}^{n} \frac{1}{\xi_j - z_j} \exp\left(-\left(\xi_j - z_j\right)^2\right) \right).
\]

Then we have \( \tilde{u} \in \tilde{\mathcal{O}}(\Omega\#K; E) \) and \( b(\tilde{u}) = u \).

**Proof** We prove this in the similar way to Theorem 3.6.4. This completes the proof. Q.E.D.