Chapter 7

Mixed Fourier hyperfunctions

In this chapter, we study the realization of mixed Fourier hyperfunctions as boundary values of slowly increasing holomorphic functions. In order to do so, we prove the Oka-Cartain-Kawai Theorem B, Dolbeault-Grothendick resolutions of $\mathcal{O}^\sharp$ and $\mathcal{O}_2$, Malgrange’s Theorem, Serre’s duality theorem, Martineau-Harvey’s Theorem and Sato’s Theorem. In this chapter, we consider the derivatives of $L^2_{\text{loc}}$, $L^2_{\text{loc}}$-functions in the sense of $L^2_{\text{loc}}$-derivatives with respect to $L^2_{\text{loc}}$-convergence. Thus, in order to study the theory of mixed Fourier hyperfunctions, we use only the method of the classical analysis.

7.1 The Oka-Cartan-Kawai Theorem B

In this section, we prove the Oka-Cartan-Kawai Theorem B for the sheaves $\mathcal{O}^\sharp$ and $\mathcal{O}_2$.

For a 2-tuple $n = (n_1, n_2)$ of two natural numbers with $|n| = n_1 + n_2 \neq 0$, we denote by $\mathbf{C}^\sharp, n$ the product space $\mathbf{C}^{n_1} \times \mathbf{C}^{n_2}$ and by $\mathbf{R}^\sharp, n$ the product space $\mathbf{R}^{n_1} \times \mathbf{R}^{n_2}$ and by $\mathbf{C}^{|n|}$ the product space $\mathbf{C}^{n_1+n_2} = \mathbf{C}^{n_1} \times \mathbf{C}^{n_2}$. We denote $z = (z', z'') \in \mathbf{C}^{|n|}$ with $z' = (z_1, \cdots, z_{n_1})$ and $z'' = (z_{n_1+1}, \cdots, z_{|n|})$. In this chapter, we assume $|n| \geq 1$.

Definition 7.1.1 (The sheaf $\mathcal{O}^\sharp$ of germs of slowly increasing holomorphic functions) We define the sheaf $\mathcal{O}^\sharp$ to be the sheafification of the presheaf $\{\mathcal{O}^\sharp(\Omega); \Omega$ is an open set in $\mathbf{C}^\sharp, n\}$. Here, the section module $\mathcal{O}^\sharp(\Omega)$ on an open set $\Omega$ in $\mathbf{C}^\sharp, n$ is the space of all holomorphic functions $f(z)$ on $\Omega \cap \mathbf{C}^{|n|}$ such that, for any positive number $\varepsilon$ and for any compact set $K$ in $\Omega$, the estimate

$$\sup\{|f(z)|e(-\varepsilon|z|); z \in K \cap \mathbf{C}^{|n|}\} < \infty$$

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Definition 7.1.2 (The sheaf $\mathcal{O}_\sharp$ of germs of rapidly decreasing holomorphic functions) We define the sheaf $\mathcal{O}_\sharp$ to be the sheafification of the presheaf $\{\mathcal{O}_\sharp(\Omega); \Omega$ is an open set in $\mathcal{C}^{\sharp, n}\}$. Here the section module $\mathcal{O}_\sharp(\Omega)$ on an open set $\Omega$ in $\mathcal{C}^{\sharp, n}$ is the space of all holomorphic functions $f(z)$ on $\Omega \cap \mathcal{C}^{\sharp, n}$ such that, for any compact set $K$ in $\Omega$, there exists some positive constant $\delta$ so that the estimate
\[
\sup\{|f(z)|e(\delta|z|); z \in K \cap \mathcal{C}^{\sharp, n}\} < \infty
\]
holds.

Definition 7.1.3 An open set $V$ in $\mathcal{C}^{\sharp, n}$ is said to be an $\mathcal{O}^{\sharp, n}$-pseudoconvex open set if it satisfies the following conditions (1) and (2):

1. $\sup\{|\text{Im} z'|, |\text{Im} z''| - |\text{Re} z''|; z = (z', z'') \in V \cap \mathcal{C}^{\sharp, n}\} < \infty$ holds.

2. There exists a $C^\infty$-plurisubharmonic function $\varphi(z)$ on $V \cap \mathcal{C}^{\sharp, n}$ having the following two properties (i) and (ii):

   (i) The closure of $V_c = \{z \in V \cap \mathcal{C}^{\sharp, n}; \varphi(z) < c\}$ in $\mathcal{C}^{\sharp, n}$ is a compact subset of $V$ for any real $c$.

   (ii) $\varphi(z)$ is bounded on $L \cap \mathcal{C}^{\sharp, n}$ for any compact set $L$ in $V$.

Then we prove the Oka-Cartan-Kawai Theorem B by the similar method to that in section 1.1.

Theorem 7.1.4 (The Oka-Cartan-Kawai Theorem B) For any $\mathcal{O}^{\sharp}$-pseudoconvex open set $V$ in $\mathcal{C}^{\sharp, n}$, we have the equalities
\[
H^s(V, \mathcal{O}^{\sharp, p}) = 0
\]
for $p \geq 0$ and $s \geq 1$.

Proof Since $V$ is paracompact, $H^s(V, \mathcal{O}^{\sharp, p})$ coincides with the Čech cohomology group. Therefore we have only to prove
\[
\lim_{U} H^s(U, \mathcal{O}^{\sharp, p}) = 0.
\]
Here $\mathcal{U} = \{U_j\}_{j \geq 1}$ is a locally finite open covering of $V$ so that $V_j = U_j \cap C^{[n]}$ is pseudoconvex for $j \geq 1$. We can chose such a covering of $V$ because $V$ is an $O^s$-pseudoconvex open set.

Now we define

$$C^s(\mathcal{Z}_{(p, q)}^s(\{V_j\}))$$

to be the set of all cochains

$$c = \{c_J; J = (j_0, j_1, \ldots, j_s) \in N^{s+1}\}$$

of forms of type $(p, q)$ satisfying the following two conditions (i) and (ii):

(i) $\bar{\partial}c_J = 0$ holds in $V_J = V_{j_0} \cap V_{j_1} \cap \cdots \cap V_{j_s}$.

(ii) For any positive $\varepsilon$ and any finite subset $M$ of $N^{s+1}$, the estimate

$$\sum_{J \in M} \int_{V_J} |c_J|^2 e(-\varepsilon \|z\|) d\lambda < \infty$$

holds.

Here $d\lambda$ is the Lebesgue measure on $C^{[n]}$ and $\|z\|$ is a convex $C^\infty$-function of $z$, which is a modification of $\sum_{j=1}^{[n]} |z_j|$.

Now we prove the following.

**Lemma 7.1.5** We assume $p, q \geq 0$. If $c \in C^s(\mathcal{Z}_{(p, q)}^s(\{V_j\}))$ satisfies the conditions $\delta c = 0$, then there exists some $c' \in C^{s-1}(\mathcal{Z}_{(p, q)}^{s-1}(\{V_j\}))$ such that $\delta c' = c$ holds. Here $\delta$ means the coboundary operator.

By virtue of Lemma 7.1.5, we obtain the theorem 7.1.4 as the special case of this Lemma 7.1.5 where $q = 0$. Here we use the Cauchy’s integral formula and we change the $L^2$-norm to the sup-norm for holomorphic functions.

**Proof of the Lemma 7.1.5** We denote by $\{\chi_j\}$ the partition of unity subordinate to $\{V_j\}$ and define $b_I = \sum_j \chi_j c_{jI}$ for $I \in N^s$. Since we have $\delta c = 0$, we have $\delta b = c$. Thus we have the equality $\delta \bar{\partial} b = 0$ by
virtue of the condition $\bar{\partial}c = 0$. Since we have $\sum \chi_j = 1$ and $\chi_j \geq 0$, we have the inequality
\[ \int_{V_I} |b_I|^2 e(-\varepsilon \|z\|) d\lambda \leq \sum_j \int_{V_I} \chi_j |c_jI|^2 e(-\varepsilon \|z\|) d\lambda \]
for any positive number $\varepsilon$ by virtue of Cauchy-Schwarz's inequality.

By the assumption of the existence of $C^\infty$-plurisubharmonic function $\varphi(z)$ in Definition 7.1.3, we find some plurisubharmonic function $\psi(z)$ on $W = V \cap C^{|n|}$ which satisfies the following two conditions (1) and (2):

(1) $\sum |\bar{\partial}\chi_j| \leq e(\psi(z))$ holds.

(2) $\sup \{\psi(z); z \in K \cap C^{|n|}\} \leq C_K$ holds, where $C_K$ is some positive constant for any compact subset $K$ of $W$.

Thus, by virtue of the condition on $c$, we have the estimate
\[ \sum_{I \in N} \int_{V_I} |\bar{\partial}b_I|^2 e(-\varepsilon \|z\| - \psi(z)) d\lambda < \infty \]
for any positive number $\varepsilon$ and any finite subset $N$ of $N^s$.

Now we consider the case $s = 1$. By virtue of the condition $\delta(\bar{\partial}b) = 0$, $\bar{\partial}b$ defines a global section $f$ on $W = V \cap C^{|n|}$. Then, by Hörmander [8], Theorem 4.4.2, p.94, we prove the existence of the solution $u$ of the equations $\bar{\partial}u = f$ such that we have the estimate
\[ \int_{K \cap C^{|n|}} |u|^2 e(-\varepsilon \|z\|)(1 + |z|^2)^{-2} d\lambda < \infty \]
for any positive number $\varepsilon$ and any compact subset $K$ of $V$.

If we define $c'_I = b_I - u|_{V_I}$, then $\bar{\partial}c'_I = 0$ and $\delta c' = \delta b = c$. Clearly we have $c' \in C^{s-1}(Z_{[p,q]}^{\text{loc}}(\{V_j\}))$.

Now we consider the case $s > 1$. In this case, we use the induction on $s$. By the induction hypotheses, there exists $b' \in C^{s-2}(Z_{[p,q]}^{\text{loc}}(\{V_j\}))$ such that $\delta b' = \bar{\partial}b$ holds holds. By virtue of Hörmander [8], Theorem 4.4.2, p.94, we have $b'' = \{b''_H\}_{H \in N^{s-1}}$ such that we have the equality $b''_H = \bar{\partial}b''_H$ and the estimate
\[ \sum_{H \in L} \int_{V_H} |b''_H|^2 e(-\varepsilon \|z\| - \psi(z))(1 + |z|^2)^{-2} d\lambda < \infty \]
for any positive number $\varepsilon$ and any finite subset $L$ of $N^{s-1}$. Therefore $c' = b - \delta b''$ satisfies all the required conditions. This completes the proof of Lemma 7.1.5. Q.E.D.

This completes the proof of the theorem. Q.E.D.

Now we prove the Malgrange’s theorem for the sheaf $\mathcal{A}^t$ of germs of slowly increasing real analytic functions. Here we define that the sheaf $\mathcal{A}^t$ is the restriction of $\mathcal{O}^t$ to $R^t \cdot n$. Namely we have $\mathcal{A}^t = \mathcal{O}^t|_{R^t \cdot n}$. Then we have the following.

**Theorem 7.1.6 (Malgrange)** For an arbitrary set $\Omega$ in $R^t \cdot n$, we have the equalities

$$H^s(\Omega, \mathcal{A}^t; p) = 0$$

for $p \geq 0$ and $s \geq 1$.

**Proof** By virute of Ito [15], Theorem 8.1.9, $\Omega$ has a fundamental system $\{\tilde{\Omega}\}$ of $\mathcal{O}^t$-pseudoconvex open neighborhoods. Then, by virtue of the Oka-Cartan-Kawai Theorem B and Schapira [56], Theorem B 42, we have the equalities

$$H^s(\Omega, \mathcal{O}^t; p) = \lim_{\tilde{\Omega} \to R^t \cdot n \rightarrow \Omega} H^s(\tilde{\Omega}, \mathcal{O}^t; p) = 0$$

for $p \geq 0$ and $s \geq 1$.

This completes the proof of the theorem. Q.E.D.

Next we prove the Oka-Cartan-Kawai Theorem B for the sheaf $\mathcal{O}^t$. We prove this by the method similar to that of Theorem 7.1.4. Thus we have the following.

**Theorem 7.1.7 (The Oka-Cartan-Kawai Theorem B)** For any $\mathcal{O}^t$-pseudoconvex open set $V$ in $C^t \cdot n$, we have the equalities

$$H^s(V, \mathcal{O}^t_p) = 0$$

for $p \geq 0$ and $s \geq 1$.

**Proof** Since $V$ is paracompact, $H^s(V, \mathcal{O}^t_p)$ coincides with the Čech cohomology group. So we have only to prove the equalities

$$\lim_{U_t} H^s(U, \mathcal{O}^t_p) = 0.$$
Here \( \mathcal{U} = \{U_j\}_{j \geq 1} \) is a locally finite open covering of \( V \) so that \( V_j = U_j \cap \mathcal{C}^{[n]} \) is pseudoconvex for \( j \geq 1 \). We can choose such a covering of \( V \) because \( V \) is an \( \mathcal{O}^\sharp \)-pseudoconvex open set.

Here we use the notation in the proof of Theorem 7.1.4.

An arbitrary element in \( H^s(\mathcal{U}, \mathcal{O}_\sharp^P) \) is defined to be a class \([d]\) with a cocycle \( d = \{d_J\} \) as its representative.

Let \([d]\) be an arbitrary element in \( H^s(\mathcal{U}, \mathcal{O}_\sharp^P) \) with a cocycle \( d = \{d_J\} \) as its representative, we define an element \( c = \{c_J\} \) in \( C^*(Z^{\sharp, \text{loc}}_{(p, 0)}(\{V_j\})) \) such as \( \delta c = 0 \) holds by putting \( c_J = d_J \cdot h_\varepsilon(z) \) for small positive \( \varepsilon \). Here \( h_\varepsilon(z) \) is defined by the formula

\[
h_\varepsilon(z) = \left( \prod_{j=1}^{n_1} \cosh(\varepsilon z_j) \right) \cdot \cosh\left( \frac{\varepsilon}{2} \sqrt{(z')^2} \right)
\]

Further \( \delta \) denotes the coboundary operator. Then we find some \( c' \in C^{s-1}(Z^{\sharp, \text{loc}}_{(p, 0)}(\{V_j\})) \) such that \( \delta c' = c \) holds. If we put \( d'_I = c_I \cdot (h_\varepsilon(z))^{-1} \), then \( d' = \{d'_I\} \) is a cochain with values in \( \mathcal{O}_\sharp \) such that \( \delta d' = d \) holds.

Thus the element \([d]\) in \( H^s(\mathcal{U}, \mathcal{O}_\sharp^P) \) with its representative \( d \) is equal to zero. Since a class \([d]\) with its representative \( d \) is an arbitrary element in \( H^s(\mathcal{U}, \mathcal{O}_\sharp^P) \), we have \( H^s(\mathcal{U}, \mathcal{O}_\sharp^P) = 0 \). This completes the proof.

Q.E.D.

At last we prove the Malgrange’s theorem for the sheaf \( \mathcal{A}_\sharp \) of germs of rapidly decreasing real analytic functions. Here we define the sheaf \( \mathcal{A}_\sharp \) to be the restriction of \( \mathcal{O}_\sharp \) to \( \mathbb{R}^\ell, n \). Namely we have \( \mathcal{A}_\sharp = \mathcal{O}_\sharp|_{\mathbb{R}^\ell, n} \).

Then we have the following.

**Theorem 7.1.8 (Malgrange)** For an arbitrary set \( \Omega \) in \( \mathbb{R}^\ell, n \), we have the equalities \( H^s(\Omega, \mathcal{A}_\sharp) = 0 \) for \( p \geq 0 \) and \( s \geq 1 \).

**Proof** We prove this by the method similar to that of Theorem 7.1.6. This completes the proof.

Q.E.D.

### 7.2 The Dolbeault-Grothendieck resolutions of the sheaves \( \mathcal{O}^\sharp \) and \( \mathcal{O}_\sharp \)

In this section, we construct the soft resolutions of the sheaves \( \mathcal{O}^\sharp \) and \( \mathcal{O}_\sharp \) and prove some of their consequences.
At first, we define the sheaf $L^\# = L^\#_{2, \text{loc}}$ of germs of slowly increasing locally $L^2$-functions over $\mathbb{C}^{\# \cdot n}$.

**Definition 7.2.1** We define the sheaf $L^\#$ to be the sheafification of the presheaf \{\(L^\#(\Omega)\); \(\Omega\) is an open set in \(\mathbb{C}^{\# \cdot n}\}\}. Here, for an open set \(\Omega\) in \(\mathbb{C}^{\# \cdot n}\), the section module \(L^\#(\Omega)\) is the space of all \(f \in L^2, \text{loc}(\Omega \cap \mathbb{C}^{\# \cdot n})\) such that, for any \(\varepsilon > 0\) and any relatively compact open subset \(\omega\) of \(\Omega\), \(e(-\varepsilon\|z\|)f(z)|_\omega\) belongs to \(L^2(\omega \cap \mathbb{C}^{\# \cdot n})\). Here \(e(-\varepsilon\|z\|)f(z)|_\omega\) denotes the restriction of \(e(-\varepsilon\|z\|)f(z)\) to \(\omega\) and \(\|z\|\) is a convex \(C^\infty\)-function, which is a modification of \(\sum_{j=1}^{[n]} |z_j|\).

Then \(L^\#\) is a soft FS*-sheaf.

Then we have the following.

**Definition 7.2.2** We assume \(p, q \geq 0\). We define the sheaf \(L^{\#, p, q} = L^{\#, p, q}_{2, \text{loc}}\) to be the sheafification of the presheaf \{\(L^{\#, p, q}(\Omega)\); \(\Omega\) is an open set in \(\mathbb{C}^{\# \cdot n}\}\}. Here, for an open set \(\Omega\) in \(\mathbb{C}^{\# \cdot n}\), the section module \(L^{\#, p, q}(\Omega)\) is the space of all \(f \in L^{\#, p, q}(\Omega) = L^{\#, p, q}_{2, \text{loc}}(\Omega)\) such that \(\bar{\partial}f \in L^{\#, p, q+1}(\Omega) = L^{\#, p, q+1}_{2, \text{loc}}(\Omega)\) holds. We put \(L^{\#} = L^{\#, 0, 0}\).

Then \(L^{\#, p, q}\) is a soft FS*-sheaf, \((p \geq 0, q \geq 0)\).

Then we have the following.

**Theorem 7.2.3 (The Dolbeault-Grothendieck resolution)** The sequence of the sheaves over \(\mathbb{C}^{\# \cdot n}\)

\[
0 \longrightarrow O^{\#, p} \longrightarrow L^{\#, p, 0} \overset{\bar{\partial}}{\longrightarrow} L^{\#, p, 1} \overset{\bar{\partial}}{\longrightarrow} \cdots \overset{\bar{\partial}}{\longrightarrow} L^{\#, p, [n]} \longrightarrow 0
\]

is exact. Here we assume \(p \geq 0\).

**Proof** For two arbitrary positive numbers \(d, d' > 0\), we choose

\[
U = \text{int}\{z \in \mathbb{C}^{[n]}; |\text{Im } z'| < d, |\text{Im } z''| - |\text{Re } z''| < d'\}^n.
\]

Here \(\text{int}(A^a)\) denotes the interior of the closure \(A^a\) of a set \(A\) in \(\mathbb{C}^{\# \cdot n}\). Then the exactness of the sequence

\[
0 \longrightarrow O^{\#, p}|_U \longrightarrow L^{\#, p, 0}|_U \overset{\bar{\partial}}{\longrightarrow} L^{\#, p, 1}|_U \longrightarrow \cdots
\]
is evident. In fact, Let $\Omega$ be a relatively compact open set in $U$. Let $u \in L^5_p,0(\Omega)$ such that $\bar{\partial}u = 0$ holds. Then, if we write $u$ in the form

$$u = \sum_{|I| = p} u_I dz_I,$$

we have

$$\frac{\partial u_I}{\partial \bar{z}_j} = 0, \quad j = 1, 2, \ldots, |n|.$$

Therefore we obtain the equation

$$\sum_{j=1}^{|n|} \frac{\partial}{\partial z_j} \frac{\partial}{\partial \bar{z}_j} u_I = 0.$$

Since the operator

$$\sum_{j=1}^{|n|} \frac{\partial}{\partial z_j} \frac{\partial}{\partial \bar{z}_j}.$$

is elliptic on $\Omega \cap R^{2|n|}$, it follows from Weyl’s Lemma that $u_I$’s are analytic on $\Omega$. Thus $u_I$’s are holomorphic. Hence we prove $u_I \in \mathcal{O}^5(\Omega)$ by virtue of the exchangeability of $L_2$-norm and the sup-norm for holomorphic functions.

Then we prove the exactness of the sequence

$$L^5_p,0|U \xrightarrow{\bar{\partial}} L^5_p,1|U \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} L^5_p,|n| |U \xrightarrow{\partial} 0.$$

For this purpose, we have only to prove the exactness of the sequence of stalks

$$L^5_p,0_z \xrightarrow{\bar{\partial}} L^5_p,1_z \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} L^5_p,|n|_z \xrightarrow{\partial} 0$$

for every $z \in U$. We prove this by virtue of Hörmander [8], Theorem 4.4.2 because every $z \in U$ has a fundamental system of $\mathcal{O}^5$-pseudoconvex open neighborhoods. This completes the proof. Q.E.D.

**Corollary 1** We use the notation in Theorem 7.2.3. For an open set $\Omega$ in $C^p_{n}$, we have the following isomorphism:

$$H^q(\Omega, \mathcal{O}^5_p) \cong \left\{ f \in L^5_p, q, \Omega; \bar{\partial}f = 0 \right\} / \left\{ \partial g; \quad g \in L^5_p, q-1(\Omega) \right\}.$$
Here we assume \( p \geq 0, \ q \geq 1 \).

**Corollary 2** Let \( \Omega \) be an \( \mathcal{O}^n \)-pseudoconvex open set in \( C^n \). Then the equation \( \overline{\partial} u = f \) has a solution \( u \in L^{p, q}_{2, \text{loc}}(\Omega) \) for every \( f \in L^{p, q+1}_{2, \text{loc}}(\Omega) \) such that \( \overline{\partial} f = 0 \) holds. Here \( p \) and \( q \) are two natural numbers.

**Proof** It follows from Theorem 7.1.4 and Corollary 1 to Theorem 7.2.3. This completes the proof. Q.E.D.

We now define the sheaf \( L_{p, 2, \text{loc}} \) of germs of rapidly decreasing locally \( L^2 \)-functions.

**Definition 7.2.4** We define the sheaf \( L^p \) to be the sheafification of the presheaf \( \{ L^p(\Omega); \ \Omega \text{ is an open set in } C^n \} \). Here, for an open set \( \Omega \) in \( C^n \), the section module \( L^p(\Omega) \) is the space of all \( f \in L^p_{2, \text{loc}}(\Omega \cap C^n) \) such that, for any relatively compact open subset \( \omega \) of \( \Omega \), there exists some positive \( \delta \) such that \( e(\delta \| z \|) f(z) \mid_{\omega \cap C^n} \in L^p_{2, \text{loc}}(\Omega) \) holds.

Then \( L^p \) is a soft FS\(^*\)-sheaf.

**Definition 7.2.5(The sheaf \( L^{p,q} \))** We assume \( p, q \geq 0 \). We define the sheaf \( L^{p,q} = L^{p,q}_{2, \text{loc}} \) to be the sheafification of the presheaf \( \{ L^{p,q}(\Omega); \ \Omega \text{ is an open set in } C^n \} \). Here, for an open set \( \Omega \) in \( C^n \), the section module \( L^{p,q}(\Omega) \) is the space of all \( f \in L^{p,q}_{2, \text{loc}}(\Omega) \) such that \( \overline{\partial} f \in L^{p,q+1}_{2, \text{loc}}(\Omega) = L^{p,q+1}_{2, \text{loc}}(\Omega) \) holds. We put \( L^p = L^{0,p} \).

Then \( L^{p,q} \) is a soft FS\(^*\)-sheaf.

Then we have the following.

**Theorem 7.2.6(The Dolbeault-Grothendick resolution)** The sequence of the sheaves over \( C^n \)

\[
0 \longrightarrow C^p_{\mathbb{R}} \longrightarrow L^{p,0}_{\mathbb{R}} \longrightarrow \overline{\partial} L^{p,1}_{\mathbb{R}} \longrightarrow \overline{\partial} L^{p,|n|}_{\mathbb{R}} \longrightarrow 0
\]

is exact. Here we assume \( p \geq 0 \).

**Proof** For two arbitrary positive numbers \( d, \ d' > 0 \), put

\[
U = \text{int}\{ z \in C^n; \ |\text{Im} \ z'| < d, \ |\text{Im} \ z''| - |\text{Re} \ z''| < d' \}.
\]
Then the exactness of the sequence

\[
0 \longrightarrow \mathcal{O}_n^p|U \longrightarrow \mathcal{L}_n^p,0|U \longrightarrow \mathcal{L}_n^p,1|U
\]

is proved by the similar way to that of Theorem 7.2.3.

Next we prove the exactness of the sequence

\[
\mathcal{L}_n^p,0|U \xrightarrow{\bar{\partial}} \mathcal{L}_n^p,1|U \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{L}_n^p,[n]|U \longrightarrow 0.
\]

Assume \(z = (z', z'') \in U\). Let \(\Omega\) be an open neighborhood of \(z\) of the form \(\Omega' \times \Omega''\), where \(\Omega'\) is an open neighborhood of \(z'\) in \(C^{n_1}\) and \(\Omega''\) is an open neighborhood of \(z''\) in \(\tilde{C}^{n_2}\) of the form \(V_{\delta, A}\) in Lemma 3.2.7 for some \(\delta\) and \(A\) such that \(0 < \delta < 1\) and \(A > 0\) hold. Let \(f\) be an element in \(\mathcal{L}_n^{p, q+1}(\Omega)\) such that \(\bar{\partial}f = 0\) holds. Then, for some \(\varepsilon > 0\), we have \(f \cdot h_\varepsilon(z) \in \mathcal{L}_n^{p, q+1}(\Omega)\). Here we put

\[
h_\varepsilon(z) = \left(\prod_{j=1}^{n_1} \cosh(\varepsilon z_j)\right) \cdot \cosh\left(\frac{\varepsilon}{2} \sqrt{(z'')^2}\right).
\]

Since \(\bar{\partial}(f \cdot h_\varepsilon(z)) = 0\) holds, we find some \(v \in \mathcal{L}_n^{p, q}(\omega)\) for some open neighborhood \(\omega = \omega' \times \omega''\) of \(z\) with \(z' \in \omega' \subset \Omega'\) and \(z'' \in \omega'' \subset \Omega''\) such that \(\bar{\partial}v = f \cdot h_\varepsilon(z)\) holds. Here we we can choose an open neighborhood \(\omega\) of \(z\) so that \(h_\varepsilon(z) \neq 0\) holds on \(\omega \cap C^n\). Then \(u = \frac{v}{h_\varepsilon(z)}\) belongs to \(\mathcal{L}_n^{p, q}(\omega)\) and \(\bar{\partial}u = f\) holds.

This completes the proof. Q.E.D.

**Corollary 1**

We use the notation in Theorem 7.2.6. For an open set \(\Omega\) in \(C^{n, n}\), we have the following isomorphism:

\[
H^p(\Omega, \mathcal{O}_n^p) \cong \left\{ f \in \mathcal{L}_n^{p, q}(\Omega); \ \bar{\partial}f = 0 \right\} / \left\{ \bar{\partial}g; \ g \in \mathcal{L}_n^{p, q-1}(\Omega) \right\}.
\]

Here we assume \(p \geq 0\) and \(q \geq 1\).

**Corollary 2**

Let \(\Omega\) be an \(\mathcal{O}^1\)-pseudoconvex open set in \(C^{n, n}\). Then the equation \(\bar{\partial}u = f\) has a solution \(u \in \mathcal{L}_n^{p, q}(\Omega)\) for every \(f \in \mathcal{L}_n^{p, q+1}(\Omega)\) such that \(\bar{\partial}f = 0\) holds. Here \(p\) and \(q\) are two natural numbers.
Proof  It follows from Theorem 7.1.7 and Corollary 1 to Theorem 7.2.6. This completes the proof. Q.E.D.

Now we construct another soft resolutions of the sheaves $\mathcal{O}^\#$ and $\mathcal{O}_z^\#$ for later applications.

At first, we give some preliminary facts. For an integer $s \geq 0$ and for an open set $\Omega$ in $C^{s, n}$, we put

$$W^\#_s(\Omega) = \{ f \in W^s_{s, \text{loc}}(\Omega \cap C^{[n]}) ; \text{ for any positive } \epsilon \text{ and for every relatively compact open set } \omega \text{ of } \Omega \text{ and every } \alpha \in N^{2|n|} \text{ with } |\alpha| \leq s, \epsilon(-\epsilon\|z\|)|f^{(\alpha)}(z)|_{\omega \cap C_j^{[n]}} \in L_2(\omega \cap C^{[n]}) \}.$$  

Then we denote by $W^{\#}_{s, p, q}(\Omega)$ the space of all differential forms of type $(p, q)$ whose coefficients in $W^\#_s(\Omega)$.

Then we have the following.

**Theorem 7.2.7** Let $\Omega$ be an $\mathcal{O}^\#$-pseudoconvex open set in $C^{s, n}$. We assume that $s$ is an integer with $0 \leq s \leq \infty$. Then the equation $\overline{\partial}u = f$ has a solution $u \in W^{\#}_{s, p+1, q}(\Omega)$ for every $f \in W^{\#}_{s, p, q+1}(\Omega)$ such that $\overline{\partial}f = 0$ holds. Every solution of the equation $\overline{\partial}u = f$ has this property in the case $q = 0$. Here we assume $p, q \geq 0$.

**Proof** (a) At first we assume that $q = 0$. By virtue of Corollary 2 to Theorem 7.2.3, the equation $\overline{\partial}u = f$ has a solution $u = \sum_1' u_I dz^I \in \mathcal{L}_{s+1, \text{loc}}^{\#}(\Omega)$ because $f \in \mathcal{L}_{s, \text{loc}}^{\#}(\Omega)$ and $\overline{\partial}f = 0$ hold. The equation $\overline{\partial}u = f$ means that

$$\frac{\partial(u_I|_{\Omega \cap C^{[n]}})}{\partial z_j} = f_I, j|_{\Omega \cap C^{[n]}} \in W_{s+1, \text{loc}}(\Omega \cap C^{[n]})$$

holds for all $I$ and $j$. Thus, by Hörmander [8], Theorem 4.2.5, we have $u_I \in W_{s+1, \text{loc}}(\Omega \cap C^{[n]})$. Then, by Nagamachi [47], Lemma 4.3, we conclude that $u_I \in W^{\#}_{s+1}(\Omega)$ holds.

(b) Next we assume that $q > 0$. Then, by Hörmander [4], Theorem 4.2.5, we find $u \in W^{p, q}_{s+1, \text{loc}}(\Omega \cap C^{[n]})$ such that $\overline{\partial}u = f$ holds. Then,
by Nagamachi [27], Lemma 4.2, we conclude that \( u \in W^{s+1, p, q}_{s+1}(\Omega) \) holds. This completes the proof. Q.E.D.

Now we define the sheaf \( \mathcal{E}^\sharp \) of germs of slowly increasing \( C^\infty \)-functions over \( C^\infty_n \).

**Definition 7.2.8** We define the sheaf \( \mathcal{E}^\sharp \) to be the sheafification of the presheaf \( \{ \mathcal{E}^\sharp(\Omega); \Omega \text{ is an open set in } C^\infty_n \} \). Here the section module \( \mathcal{E}^\sharp(\Omega) \) on an open set \( \Omega \) in \( C^\infty_n \) is defined as follows:

\[
\mathcal{E}^\sharp(\Omega) = \{ f \in \mathcal{E}(\Omega \cap C^{[n]}); \text{for any positive } \varepsilon \text{ and any compact set } K \text{ in } \Omega \text{ and any } \alpha \in N^{2[n]}, \text{ the estimate } \sup\{|f^{(\alpha)}(z)|e(-\varepsilon|z|); z \in K \cap C^{[n]}\} < \infty \text{ holds} \}.
\]

Then \( \mathcal{E}^\sharp \) is a soft nuclear Fréchet sheaf.

Then we have the following.

**Theorem 7.2.9** Let \( \Omega \) be an \( \mathcal{O}^\infty \)-pseudoconvex open set in \( C^\infty_n \). Then the equation \( \bar{\partial}u = f \) has a solution \( u \in \mathcal{E}^{\sharp, p, q}(\Omega) \) for every \( f \in \mathcal{E}^{\sharp, p, q+1}(\Omega) \) such that \( \bar{\partial}f = 0 \) holds. Every solution of the equation \( \bar{\partial}u = f \) has this property in the case \( q = 0 \). Here we assume \( p, q \geq 0 \).

**Proof** Since \( f \in W^{s, p, q+1}_{s+1}(\Omega) \) holds for every natural number \( s \), we see that \( u \in W^{s+1, p, q}_{s+1}(\Omega) \) holds for every natural number \( s \). But, by the Sobolev lemma, we have the inclusion relation

\[
W^{s, p, q}_{s+2[n]}(\Omega) \subset C^{\sharp, p, q}_{s}(\Omega).
\]

Here we put

\[
C^{\sharp}_{s}(\Omega) = \{ f \in C^s(\Omega \cap C^{[n]}); \text{for any positive } \varepsilon \text{ and any compact set } K \text{ in } \Omega \text{ and any } \alpha \in N^{[n]} \text{ with } |\alpha| \leq s, \text{ the estimate } \sup\{|f^{(\alpha)}(z)|e(-\varepsilon|z|); z \in K \cap C^{[n]}\} < \infty \text{ holds} \}.
\]
Thus we have $u \in \mathcal{E}^{p, q}(\omega)$. This completes the proof. Q.E.D.

Then we have the following.

**Theorem 7.2.10 (The Dolbeault-Grothendieck resolution)** The sequence of the sheaves over $\mathbb{C}^{\#}$:

$$
0 \longrightarrow \mathcal{O}^{p, 0} \longrightarrow \mathcal{E}^{p, 0} \longrightarrow \mathcal{E}^{p, 1} \longrightarrow \cdots \longrightarrow \mathcal{E}^{p, |n|} \longrightarrow 0
$$

is exact. Here we assume $p, q \geq 0$.

**Proof** It follows immediately from Theorem 7.2.9. This completes the proof. Q.E.D.

**Corollary** We use the notation in Theorem 7.2.10. For an open set $\Omega$ in $\mathbb{C}^{\#}$, we have the following isomorphism:

$$
H^q(\Omega, \mathcal{O}^{p}) \cong \{f \in \mathcal{E}^{p, q}(\Omega); \overline{\partial}f = 0\}
$$

Here we assume $p \geq 0$ and $q \geq 1$.

Now we define the sheaf $\mathcal{E}_t$ of rapidly decreasing $C^\infty$-functions over $\mathbb{C}^{\#}$.

**Definition 7.2.11** We define the sheaf $\mathcal{E}_t$ to be the sheafification of the presheaf $\{\mathcal{E}_t(\Omega); \Omega$ is an open set in $\mathbb{C}^{\#}\}$. Here, the section module $\mathcal{E}_t(\Omega)$ on an open set $\Omega$ in $\mathbb{C}^{\#}$ is the space of all $C^\infty$-functions on $\Omega \cap \mathbb{C}^{[n]}$ such that, for any compact set $K$ in $\Omega$ and any $\alpha \in \mathbb{N}^{2|n|}$, there exists some positive constant $\delta$ such that the estimate

$$
\sup\{|f^{(\alpha)}(z)|\epsilon(\delta|z|); z \in K \cap \mathbb{C}^{[n]}\} < \infty
$$

holds.

Then $\mathcal{E}_t$ becomes a soft nuclear Fréchet sheaf.

Then we have the following.
Theorem 7.2.12 (The Dolbeault-Grothendieck resolution) The sequence of the sheaves over \( C^n \)

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \mathcal{O}_z^p & \rightarrow & \mathcal{E}_z^{p,0} & \rightarrow & \mathcal{E}_z^{p,0} & \rightarrow & \mathcal{E}_z^{p,|n|} & \rightarrow & 0
\end{array}
\]

is exact. Here we assume \( p \geq 0 \).

Proof For two arbitrary positive numbers \( d, d' > 0 \), we choose \( U = \text{int} \{ z \in C^n | \text{Im} z' < d, \text{Re} z'' < d' \} \) in \( C^n \). Let \( z = (z', z'') \in U \) and \( \Omega \) an open neighborhood of \( z \) of the form \( \Omega' \times \Omega'' \), where \( \Omega' \) is an open neighborhood of \( z' \) in \( C^{n_1} \) and \( \Omega'' \) is an open neighborhood of \( z'' \) in \( C^{n_2} \) of the form \( V_{\delta, A} \) in Lemma 3.2.7 for some \( \delta \) and \( A \) such as \( 0 < \delta < 1 \) and \( A > 0 \) hold. Let \( f \) be an element in \( \mathcal{E}_z^{p, q+1}(\Omega) \) such that \( \overline{\partial} f = 0 \) holds. Then, for some \( \varepsilon > 0 \), we have \( f \cdot h_{\varepsilon}(z) \in \mathcal{E}_z^{p, q+1}(\Omega) \).

Here we put

\[
h_{\varepsilon}(z) = \prod_{j=1}^{n_1} (\cosh(\varepsilon z_j)) \cdot \cosh(\frac{\varepsilon}{2} \sqrt{(z''^2)}).
\]

Since \( \overline{\partial}(f \cdot h_{\varepsilon}(z)) = 0 \) holds, we find some \( v \in \mathcal{E}_z^{p, q}(\omega) \) for some open neighborhood \( \omega = \omega' \times \omega'' \) of \( z \) such that \( \overline{\partial} v = f \cdot h_{\varepsilon}(z) \) holds. Here we can choose an open neighborhood \( \omega \) of \( z \) so that \( h_{\varepsilon}(z) \neq 0 \) holds on \( \omega \cap C^n \). Then \( u = \frac{v}{h_{\varepsilon}(z)} \) belongs to \( \mathcal{E}_z^{p, q}(\omega) \) and \( \overline{\partial} u = f \) holds. This completes the proof.

Q.E.D.

Corollary 1 We use the notation in Theorem 7.2.12. For an open set \( \Omega \) in \( C^n \), we have the following isomorphism:

\[
H^q(\Omega, \mathcal{O}_z^p) \cong \left\{ f \in \mathcal{E}_z^{p,q}(\Omega); \overline{\partial} f = 0 \right\} / \left\{ \overline{\partial} g; g \in \mathcal{E}_z^{p,q-1}(\Omega) \right\}.
\]

Here we assume \( p \geq 0 \) and \( q \geq 1 \).

Corollary 2 Let \( \Omega \) be an \( \mathcal{O}^k \)-pseudoconvex open set in \( C^n \). Then the equation \( \overline{\partial} u = f \) has solution \( u \in \mathcal{E}_z^{p,q}(\Omega) \) for every \( f \in \mathcal{E}_z^{p,q+1}(\Omega) \) such that \( \overline{\partial} f = 0 \) holds. Here \( p \) and \( q \) are two natural numbers.
Proof. It follows from Theorem 7.1.7 and Corollary 1 to Theorem 7.2.12. Q.E.D.

7.3 Malgrange’s Theorem

In this section, we prove the Malgrange’s Theorem.

**Theorem 7.3.1 (Malgrange’s Theorem)** Let $\Omega$ be an open set in $C^\leftarrow,n$ such that, for any $z \in \Omega \cap C^{[n]}$, $|\text{Im } z''| - |\text{Re } z''| < d$ holds for some positive constant $d > 0$ independent of $z \in \Omega \cap C^{[n]}$. Then we have the equality $H^{|n|}(\Omega, \mathcal{O}_2) = 0$.

**Proof** By virtue of Corollary 1 to Theorem 7.2.3, we have only to prove the exactness of the sequence

$$L^0, 0; |n|\rightarrow -1(\Omega) \longrightarrow L^0, 0; |n|(\Omega) \longrightarrow 0$$

in the notation of Theorem 7.2.3. But, in order to do so, we have only to prove the injectiveness and the closedrangeness of $-\partial = (\partial)'$ in the dual sequence

$$L^0, 0; \mathcal{L}_{+}^{-} \longrightarrow L^0, 0; \mathcal{L}_{+}^{0, 0} \longrightarrow 0$$

in the notation of Theorem 7.2.6 by virtue of the Serre-Komatsu duality theorem for FS$^*$-spaces. Here $L^0, q; \mathcal{L}_{+}^{0, 0}(\Omega)$ denotes the the space of sections with compact suport of $L^0, q; \mathcal{L}_{+}^{0, 0}$ on $\Omega$. We prove this by the similar method to that of Kawai [37], Theorem 3.1.8. Q.E.D.

**Corollary** Flabby dim $\mathcal{O}_2 \leq |n|$ holds.

7.4 Serre’s duality theorem

In this section, we prove the Serre’s duality theorem.

**Theorem 7.4.1** Let $\Omega$ be an open set such as in Theorem 7.3.1 and assume that $\dim H^p(\Omega, \mathcal{O}_2) < \infty$ holds for $p \geq 1$. Then we have the topological isomorphisms

$$[H^p(\Omega, \mathcal{O}_2)]' \cong H^{|n|-p}(\Omega, \mathcal{O}_2)$$
for $0 \leq p \leq |n|$.

**Proof** By virtue of Corollary 1 to Theorem 7.2.3 and Corollary 1 to Theorem 7.2.6, the cohomology groups $H^p(\Omega, \mathcal{O}^\sharp)$ and $H^{|n|}_c[-p](\Omega, \mathcal{O}_z)$ are the cohomology groups of the dual complexes

$$
\begin{align*}
0 \longrightarrow L^{0,0}_{c,0}(\Omega) & \overset{\partial}{\longrightarrow} L^{0,1}_{c,0}(\Omega) & \overset{\partial}{\longrightarrow} (1) \\
0 \longleftarrow L^{0,|n|}(\Omega) & \overset{\partial}{\longleftarrow} L^{0,|n|-1}_{c,0}(\Omega) & \overset{\partial}{\longleftarrow} (2)
\end{align*}
$$

respectively. Here the upper complex is composed of FS*-spaces and the lower complex is composed of DFS*-spaces. Since the ranges of operators $\partial$ in the upper complex are all closed by virtue of Schwartz’s Lemma, the ranges of operators $-\partial = (\partial)'$ in the lower complex are also all closed. As for this, we refer to Komatsu [38]. Hence we have the topological isomorphism

$$
[H^p(\Omega, \mathcal{O}^\sharp)]' \cong H^{|n|-p}_c(\Omega, \mathcal{O}_z)
$$

by virtue of Serre’s Lemma. As for this, we refer to Komatsu [38].

Q.E.D.

### 7.5 Martineau-Harvey’s Theorem

In this section, we prove the Martineau-Harvey’s Theorem.

**Theorem 7.5.1** Let $K$ be a compact set in $\mathcal{C}^5_{\ast,n}$ such that it has an $\mathcal{O}^\sharp$-pseudoconvex open neighborhood $\Omega$ and satisfies the conditions $H^p(K, \mathcal{O}_z) = 0$ for $p \geq 1$. Then we have the following (1) ~ (3):

1. We have the equalities
   
   $$
   H^p_K(\Omega, \mathcal{O}^\sharp) = 0
   $$

   for $p \neq |n|$.
In the case $n \geq 2$, we have the topological isomorphisms

$$H^n_K(\Omega, \mathcal{O}^\sharp) \cong H^{n-1}(\Omega \setminus K, \mathcal{O}^\sharp) \cong \mathcal{O}_t(K)'.$$ 

(3) In the case $n = 1$, we have the topological isomorphisms

$$H^1_K(\Omega, \mathcal{O}^\sharp) \cong \frac{\mathcal{O}^\sharp(\Omega \setminus K)}{\mathcal{O}^\sharp(\Omega)} \cong \mathcal{O}_t(K).$$

**Remark** If a compact set $K$ in $\mathcal{C}^\sharp, n$ has a fundamental system of $\mathcal{O}^\sharp$-pseudoconvex open neighborhoods, it satisfies the assumptions in Theorem 7.5.1.

**Proof** We prove this in the similar way to that of Theorem 3.5.1. Q.E.D.

### 7.6 Sato’s Theorem

In this section, we prove the pure-codimensionality of $\mathcal{R}^\sharp, n$ with respect to the sheaf $\mathcal{O}^\sharp$. Then we realize mixed Fourier hyperfunctions as “boundary values” of slowly increasing holomorphic functions or as relative cohomology classes of slowly increasing holomorphic functions.

**Theorem 7.6.1 (Sato’s Theorem)** Let $\Omega$ be an open set in $\mathcal{R}^\sharp, n$ and $V$ an open set in $\mathcal{C}^\sharp, n$ which contains $\Omega$ as its closed subsets. Then we have the following (1) \sim (3).

1. **The relative cohomology groups $H^p_\Omega(V, \mathcal{O}^\sharp)$ are zero for $p \neq |n|$.**
2. **The presheaf over $\mathcal{R}^\sharp, n$**

$$\Omega \longrightarrow H^{|n|}_\Omega(V, \mathcal{O}^\sharp)$$

is a flabby sheaf.

3. **This sheaf (2) is isomorphic to the sheaf $\mathcal{B}^\sharp$ of mixed Fourier hyperfunctions defined in Definition 8.3.3 of the Part I of this book.**
Proof (1) We prove this in the similar way to Kawai [37], p.482.
(2) By Malgrange’s Theorem, we conclude that flabby \( \dim O^\sharp \leq |n| \) holds. Thus, by (1) and the theorem II.3.24 of Komatsu [39], we have the conclusion.
(3) Consider the following exact sequence of relative cohomology groups

\[
0 \longrightarrow H^0_{\partial \Omega}(V, O^\sharp) \longrightarrow H^0_{\Omega^p}(V, O^\sharp) \longrightarrow H^0_\Omega(V, O^\sharp) \\
\longrightarrow H^1_{\partial \Omega}(V, O^\sharp) \longrightarrow \cdots \longrightarrow H^{|n|}_\Omega(V, O^\sharp) \\
\longrightarrow H^{|n|+1}_{\partial \Omega}(V, O^\sharp) \longrightarrow \cdots .
\]

Then, by (1) and the Martineau-Harvey’s Theorem, we have the equalities

\[
H^{|n|}_\Omega(V, O^\sharp) = 0, \quad H^{|n|+1}_{\partial \Omega}(V, O^\sharp) = 0.
\]

Thus we have the exact sequence

\[
0 \longrightarrow H^{|n|}_{\partial \Omega}(V, O^\sharp) \longrightarrow H^{|n|}_{\Omega^p}(V, O^\sharp) \longrightarrow H^{|n|}_\Omega(V, O^\sharp) \\
\longrightarrow 0.
\]

Since, by the Martineau-Harvey’s Theorem, we have the topological isomorphisms

\[
H^{|n|}_{\partial \Omega}(V, O^\sharp) \cong A_\sharp(\partial \Omega)' , \quad H^{|n|}_{\Omega^p}(V, O^\sharp) \cong A_\sharp(\Omega^p)',
\]

we obtain the algebraic isomorphism

\[
H^{|n|}_\Omega(V, O^\sharp) \cong \frac{A_\sharp(\Omega^p)'}{A_\sharp(\partial \Omega)'} = B^\sharp(\Omega).
\]

Thus the sheaf \( \Omega \rightarrow H^{|n|}_\Omega(V, O^\sharp) \) is isomorphic to the sheaf \( B^\sharp \) of mixed Fourier hyperfunctions over \( R^{\sharp n} \).

Q.E.D.

Let \( \Omega \) be an open set in \( R^{\sharp n} \). Then there exists an \( O^\sharp \)-pseudoconvex open neighborhood \( V \) of \( \Omega \) such that \( V \cap R^{\sharp n} = \Omega \) holds. As for this, we refer to Ito [15], Theorem 8.1.9. We put

\[
V_0 = V \text{ and } V_j = V \setminus \{ z \in V; \, \text{Im} \, z_j = 0 \}^a, \quad (j = 1, 2, \cdots, |n|).
\]
Then \( \mathcal{U} = \{ V_0, V_1, \ldots, V_{|n|} \} \) and \( \mathcal{U}' = \{ V_1, \ldots, V_{|n|} \} \) cover \( V \) and \( V \setminus \Omega \) respectively. Since \( V_j \) and their intersections are \( \mathcal{O}^\sharp \)-pseudoconvex open sets, the covering \( (\mathcal{U}, \mathcal{U}') \) satisfies the conditions of Leray’s Theorem. As for this, we refer to Komatsu [23]. Thus, by Leray’s Theorem, we obtain the isomorphism

\[
H^{[n]}_{\Omega}(V, \mathcal{O}^\sharp) \cong H^{[n]}(\mathcal{U}, \mathcal{U}', \mathcal{O}^\sharp).
\]

Since the covering \( \mathcal{U} \) is composed of only \(|n| + 1\) open sets \( V_j, (j = 0, 1, \ldots, |n|) \), we obtain the algebraic isomorphisms

\[
Z^{[n]}(V, V', \mathcal{O}^\sharp) \cong \mathcal{O}^\sharp(\bigcap_j V_j),
\]

\[
C^{[n]-1}(V, V', \mathcal{O}^\sharp) \cong \bigoplus_{j=1}^{[n]} \mathcal{O}^\sharp(\bigcap_{i \neq j} V_i).
\]

Hence we have the algebraic isomorphism

\[
\delta C^{[n]-1}(V, V', \mathcal{O}^\sharp) \cong \sum_{j=1}^{[n]} \mathcal{O}^\sharp(\bigcap_{i \neq j} V_i | V_1 \cap \cdots \cap V_{[n]}).
\]

Thus we have the algebraic isomorphisms

\[
H^{[n]}_{\Omega}(V, \mathcal{O}^\sharp) \cong H^{[n]}(V, V', \mathcal{O}^\sharp)
\]

\[
\cong \frac{Z^{[n]}(V, V', \mathcal{O}^\sharp)}{\delta C^{[n]-1}(V, V', \mathcal{O}^\sharp)} \cong \frac{\mathcal{O}^\sharp(\bigcap_j V_j)}{\sum_{j=1}^{[n]} \mathcal{O}^\sharp(\bigcap_{i \neq j} V_i)}.
\]

Thus we have the following.

**Theorem 7.6.2** We use the notation in the above. Then we have the algebraic isomorphisms

\[
H^{[n]}_{\Omega}(V, \mathcal{O}^\sharp) \cong H^{[n]}(V, V', \mathcal{O}^\sharp) \cong \frac{\mathcal{O}^\sharp(\bigcap_j V_j)}{\sum_{j=1}^{[n]} \mathcal{O}^\sharp(\bigcap_{i \neq j} V_i)}.
\]
By virtue of Theorem 7.6.1 and Theorem 7.6.2, we realize the mixed Fourier hyperfunctions as boundary values of slowly increasing holomorphic functions or as relative cohomology classes with coefficients in $\mathcal{O}^\#$. Thus we prove that two realizations of mixed Fourier hyperfunctions as the classes of mixed Fourier analytic functionals and as the boundary values of slowly increasing holomorphic functions are constructed independently and they are equivalent.

At last we realize mixed Fourier analytic functionals with certain compact carrier as relative cohomology classes with coefficients in $\mathcal{O}^\#$.

Let $K$ be a compact set in $C^{\#}$ of the form $K = K_1 \times \cdots \times K_{|n|}$ with compact sets $K_1, K_2, \ldots, K_{|n|}$ in $\overline{C}$ and compact sets $K_{n_1+1}, K_{n_1+2}, \ldots, K_{|n|}$ in $\overline{C}$.

Assume that $K$ admits a fundamental system of $\mathcal{O}^\#$-pseudoconvex open neighborhoods.

Then we have the equalities

$$H^p(K, \mathcal{O}^\#) = 0$$

for $p > 0$.

By virtue of the Martineau-Harvey’s Theorem, there exists the topological isomorphism

$$\mathcal{O}_\mathcal{L}(K)^\prime \cong H^{|n|}_K(\Omega, \mathcal{O}^\#).$$

Here $\Omega$ denotes an open neighborhood of $K$. Further assume that there exists an $\mathcal{O}^\#$-pseudoconvex open neighborhood $\Omega$ of $K$ such that

$$\Omega_j = \Omega \setminus \{z \in C^{[n]}; z_j \in K_j \cap C\}^a$$

is also an $\mathcal{O}^\#$-pseudoconvex open set for $j = 1, 2, \ldots, |n|$. Then $\mathcal{U} = \{\Omega_0, \Omega_1, \ldots, \Omega_{|n|}\}$ and $\mathcal{U}' = \{\Omega_1, \ldots, \Omega_{|n|}\}$ form the acyclic coverings of $\Omega$ and $\Omega \setminus K$. Here we put $\Omega_0 = \Omega$. Further set

$$\Omega^\sharp K = \bigcap_{j=1}^{|n|} \Omega_j, \quad \Omega^j = \bigcap_{i \neq j} \Omega_i, \quad (1 \leq j \leq |n|).$$

Let $\sum_j \mathcal{O}^\sharp(\Omega^j)$ be the image in $\mathcal{O}^\sharp(\Omega^\sharp K)$ of $\prod_{j=1}^{|n|} \mathcal{O}^\sharp(\Omega^j)$ by the mapping

$$(f_j)_{j=1}^{|n|} \mapsto \sum_{j=1}^{|n|} (-1)^{j+1} f_j^\prime$$

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where \( f'_j \) denotes the restriction of \( f_j \) to \( \Omega \sharp K \).

Then, by the similar method to that of Theorem 7.6.2, we have the following.

**Theorem 7.6.3** We use the notation in the above. Then we have the topological isomorphisms

\[
\mathcal{O}_K'(K) \cong H^1_K(\Omega, \mathcal{O}) \cong H^1(\mathcal{V}, \mathcal{V}', \mathcal{O}) \cong \sum_j \mathcal{O}(\Omega \sharp j).
\]